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From quantum to classical by numbers

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Abstract

We follow Kofler and Brukner (2007 *Phys. Rev. Lett.* **99** 180403) in studying the conditions under which a classical picture emerges from the results of not too accurate quantum measurements made on large macroscopic objects. We show that for such objects, consisting of a large number of microscopic elements obeying quantum laws, the Central Limit theorem guarantees the existence of classical values for collective variables, even if the corresponding operators do not commute. Owing to localisation of the overall wave function in any chosen representation, these values can be measured to a small relative error without significantly altering the state of the object. We study a simple model, which includes a rudimentary observer capable of detecting in the coordinate space the position of a macroscopic pointer. The pointer can be employed to measure such quantities, not directly accessible to the observer, as linear or angular momenta. A purely classical picture arises provided the measurements are made on macroscopic objects. Results of measurements, made on small quantum objects, cannot be predicted with certainty, but acquire certain objectivity when encoded in macroscopic pointers' positions accessible to all observers. Our estimates show that the classical conditions could, in principle, be realised for systems with number of constituent parts of the order of the Avogadro constant. It is possible that the approach captures the essential features of the quantum-to-classical behaviour, although its extension to more realistic systems is likely to be required.

1. Introduction

The old question of how, and where, quantum properties of the micro-world turn into classical-like experiences of an experimentalist continues to date without a definite answer. In quantum physics, one can observe certain outcomes whose probabilities are supplied by the theory. Calculation of these probabilities (frequencies) requires a recourse to complex valued probability amplitudes, or wave functions, whose precise status is still debated in the literature (see, for example, [1]). Seen by some as a purely mathematical tool, the amplitudes are pervasive in the theory, and their values can sometimes be divulged from the observed frequencies [2]. Moreover, measurement of a particular quantity perturbs the measured system, so that two quantities, whose operators do not commute, cannot have well defined simultaneous values.

Classical physics, by its very nature a limiting case of quantum theory, knows nothing of the mentioned difficulties. It postulates a unique verifiable outcome for each observation and, since classical observations do not disturb the monitored system, ascribes definite simultaneous values to all quantities.

Different schools approach the problem from different perspectives. For example, the standard Copenhagen interpretation of quantum mechanics (for a review see [3], and references therein) relies on the 'Heisenberg cut', a hypothetical interface between quantum events and the Observer's information. The consistent histories approach (see [4] and references therein) purports to define probabilities for a closed quantum system, and reserves no special role for an Observer. Observer's role is at best passive [5] in the Everett's many-world interpretation ([6], and references therein). Similarly, Bohmian quantum mechanics denies the Observer any role in the formulation of the physical picture of the world (see [7], and, more relevant to our discussion, [8]). Detailed arguments for and against these suggestions are beyond the scope of this paper, and can be found

elsewhere in the literature. It is fair, however, to say that none of the above approaches have yet provided a definite answer to the quantum-to-classical question.

In 2007 Kofler and Brukner demonstrated that macro realism and the classical laws of motion emerge from the standard quantum formalism, provided the measurements, made on large systems, are coarse-grained, i.e. limited in their accuracy [9]. Their work was preceded by the one of Lloyd and Slotine, who showed that imprecise (weak) measurements made on a set of identical systems can determine the properties of an individual system while affecting it only slightly [10]. It is therefore plausible, that a classical picture could emerge from quantum description, provided the Observer limits him/her/itself to studying only a certain class of classical phenomena, using only certain types of classical instruments. Let us mention that measurements and their classical limit have also been discussed in several works, see for instance [11, 12] including semiclassical approaches that show that the fine structure of quantum probability distributions cannot be resolved for a macroscopic object and can therefore be discarded [13, 14].

In this paper we will follow both of the previous references in looking for a way to recover the classical picture from quantum properties, by analysing the observation (measurement) procedures offered by quantum mechanics. More precisely, the classicality we want to study relies in the following assumptions:

(A) Provided a system is large enough, there exist measurements yielding a unique classical value, if not exactly, then with a vanishing relative error.

(B) All macroscopic quantities should have well defined classical values, in the sense of the above.

(C) The disturbance produced on the measured system should be if not exactly zero, then small enough for the next measurement of any different quantity to produce its own unique classical result.

(D) The result of any measurement can be encoded into the spatial position of a classical pointer, which can be 'read' by any number of Observers, without altering its state, or altering it only by a negligible amount.

With this in mind, we will carefully examine different situations involving a quantum system or systems, one or several quantum pointers and, eventually, rudimentary external Observers. In particular, we will evoke a simple model similar to that used in [9, 10], and study what happens if an Observer restricts him/her/itself to monitoring only large conglomerates of non-interacting elementary quantum systems, by means of instruments, whose pointers are composites of quantum particles considered in the coordinate representation. We will also consider the case where an Observer can access the information about an elementary quantum system, encoded into the position of a large macroscopic pointer. In most cases we evaluate the change, produced by an observation on the system's state, and estimate its effect on the result on a subsequent measurement of a different variable.

A more detailed layout of the paper as follows. In section 2 we briefly review the basic elements of Quantum measurement theory, used throughout this work. In section 3 we look at the classical picture, which emerges when a large set of equally polarised spin-1/2 particles, is monitored by an equally large set of quantum pointers. In section 4 we apply the Central Limit theorem (CLT) to measurements of collective additive quantities, and relate the emergence of 'classical values', to the localisation of the composite's wave function in the chosen representation. As an illustration, in section 5 we examine the case in which a large number of spin-1/2 particles are monitored by a single quantum pointer. We evaluate the damage to the composite's quantum state done by a measurement, and its consequence for a follow up measurement of a different total spin's component. Section 6 describes a similar study, this time of a large number of non-relativistic quantum particles, prepared in the same state. A classical picture is recovered for measurements resolutions, which ensure sufficiently accurate simultaneous values of the composite's centre of mass (COM) and its total momentum. In section 7 we let the cloud of particles be split after a collision with a potential barrier, and obtain a similar classical picture for its transmitted part only. In section 8 we add a 'rudimentary Observer', who's primitive sensor allows him/her/it detect ('see', see the assumption D of the Introduction) position of the COM of a large cloud of particles, which plays the role of a classical pointer. The sensor is taken to be a single pointer, capable of coupling to macroscopic objects, and whose displacement encodes the value of the measured quantity. We need to model the Observer of the system-device-observer sequence in one way or another, and this way is certainly the simplest. In section 9 such a pointer is used to perform a measurement of a component of a single spin-1/2. In section 10 we ask what would be the Observer's experience, after looking at a classical pointer, previously prepared in a superposition of spatially separated macroscopic states. In section 11 we recover the classical limit of the measurement theory, by considering a large angular momentum, monitored by a macroscopic pointer. In section 12 we look for observations, which might convince an Observer that the classical picture, hitherto perceived, is, after all, only an approximate one. We present our conclusions in section 13. Appendices A and B discuss certain technical points.

2. Quantum measurements

Our main tool will be a standard von Neumann measurement [15–17], which we will now briefly review. Suppose we set out to measure an operator $\hat{A}$, with eigenstates $|n\rangle$, and eigenvalues A^n, in a Hilbert space of dimension N. If only $J \leqslant N$ eigenvalues A_j are distinct, we can write

$$\hat{A} = \sum_{n=1}^{N} |n\rangle A^n \langle n| = \sum_{j=1}^{J} A_j \hat{\pi}_j, \quad \hat{\pi}_j \hat{\pi}_{j'} = \hat{\pi}_j \delta_{jj'}, \tag{1}$$

where $\hat{\pi}_j$ projects onto the subspace corresponding to an A_j. At $t = 0$, we couple the system in a state $|\Psi_0\rangle$ to a pointer, a massive particle with position f and momentum λ, via a brief yet strong interaction (we put $\hbar = 1$),

$$\hat{H}_{\rm int} = g\hat{\lambda}\hat{A}\delta(t) \tag{2}$$

(g being the coupling strength), and determine the pointer's position (reading) once the interaction is over. The initial state of the composite system+pointer is a product $|\Phi_0\rangle = |\Psi_0\rangle|G\rangle$, $|G\rangle = \int {\rm d}f G(f)|f\rangle$. In the following we will consider $G(f)$ a real function, peaked around $f = 0$, where its width, Δf, will determine the resolution of the measurement. It is convenient to write the system's state $|\Psi_0\rangle$ as a superposition of orthonormal states $|j\rangle$, $j = 1, 2, .., J$, $\langle j'|j\rangle = \delta_{jj'}$, corresponding to the values A_j,

$$\begin{aligned} |\Psi_0\rangle &= \sum_{j=1}^{J} \langle \Psi_0|\hat{\pi}_j|\Psi_0\rangle^{1/2}|j\rangle, \\ |j\rangle &\equiv \langle \Psi_0|\hat{\pi}_j|\Psi_0\rangle^{-1/2}\hat{\pi}_j|\Psi_0\rangle, \quad \hat{A}|j\rangle = A_j|j\rangle. \end{aligned} \tag{3}$$

This form offers several advantages. Firstly, it is easy to check that the state of the composite system+pointer immediately after the interaction, $|\Phi_1\rangle$, and the resulting probability distribution of the pointer's readings, $\rho(f)$, take particularly simple forms

$$\langle f|\Phi_1\rangle = \sum_{j=1}^{J} G(f - A_j)\langle \Psi_0|\hat{\pi}_j|\Psi_0\rangle^{1/2}|j\rangle, \; \rho(f) = \langle \Phi_1|f\rangle\langle f|\Phi_1\rangle = \sum_{j=1}^{J} G^2(f - A_j)\langle \Psi_0|\hat{\pi}_j|\Psi_0\rangle, \tag{4}$$

where we have chosen the units so as to put the coupling strength g in equation (2) to unity.

Secondly, it helps to visualise the damage which a measurement does to the state of the measured system. Indeed, if the pointer reads f, the state $\langle f|\Phi_1\rangle$ differs from $|\Psi_0\rangle$ only if the factors $G(f - A_j)$ differ between themselves. Having all G's identical, e.g. $G(f - A_j) \approx G(f)$, would only result in appearance of an unimportant overall factor $\langle f|\Phi_1\rangle = G(f)|\Psi_0\rangle$. Thirdly, writing

$$\hat{\pi}_j = \sum_{n=1}^{N} |n\rangle \Delta(A_j - A^n)\langle n|, \tag{5}$$

where $\Delta(X - Y) = 1$ if $X = Y$, and 0 otherwise, we note that in equations (4), the factor

$$\langle \Psi_0|\hat{\pi}_j|\Psi_0\rangle = \sum_{n=1}^{N} |\langle n|\Psi_0\rangle|^2 \Delta(A_j - A^n) \tag{6}$$

is just the total probability of finding the result A_j in an ideally accurate measurement of $\hat{A}$, $|G(f)|^2 \to \delta(f)$. This will let us apply the CLT [18] in the most interesting for us case of a measurement made on a large set of identical quantum systems.

The results are readily generalised to measuring an operator with a continuous spectrum (see appendix A), where we have ($\delta(z)$ is the Dirac delta)

$$\hat{A} = \int {\rm d}\nu |\nu\rangle A^\nu \langle \nu|, \quad \langle \nu|\nu'\rangle = \delta(\nu - \nu'), \tag{7}$$

$$\begin{aligned} \langle f|\Phi_1\rangle &= \int {\rm d}a \; G(f - a)\langle \Psi_0|\hat{\pi}(a)|\Psi_0\rangle^{1/2}|a\rangle, \\ \hat{\pi}(a) &= \int {\rm d}\nu |\nu\rangle \delta(A_\nu - a)\langle \nu|, \end{aligned} \tag{8}$$

and

$$\begin{aligned} |a\rangle &\equiv \langle \Psi_0|\hat{\pi}(a)|\Psi_0\rangle^{-1/2}\hat{\pi}(a)|\Psi_0\rangle, \\ \langle a'|a\rangle &= \delta(a - a'), \quad \hat{A}|a\rangle = a|a\rangle. \end{aligned} \tag{9}$$

Finally, the results of this section are easily generalised to a system, initially in a mixed state given by a convex sum of one-dimensional projectors. We would only need to apply the above analysis to each term, and then add the results, as appropriate.

2.1. A follow up measurement

Immediately after obtaining a reading f for the operator $\hat{A}$ in (7) we may decide to measure a different operator

$$\hat{B} = \int \mathrm{d}\mu |\mu\rangle B^{\mu} \langle\mu|, \tag{10}$$

using a second pointer with position f', prepared in a state $|G'\rangle$. For the final state of the system + two pointers, $|\Phi_2\rangle$, we find

$$\langle f'|\langle f|\Phi_2\rangle = \int \mathrm{d}b\ \mathrm{d}a\ G(f'-b)G(f-a)\hat{\pi}(b)\hat{\pi}(a)|\Psi_0\rangle, \tag{11}$$

with $\hat{\pi}(b) \equiv \int \mathrm{d}\mu |\mu\rangle \delta(B_{\mu} - b)\langle\mu|$. Now the joint probability distribution for the readings f and f' is given by

$$\begin{aligned}\rho(f,f') = \langle\Phi_2|f\rangle|f'\rangle\langle f'|\langle f|\Phi_2\rangle = \int \mathrm{d}b\ G'^2(f'-b) \\ \times \int \mathrm{d}a\ \mathrm{d}a'\ G^*(f-a')G(f-a)\langle\Psi_0|\hat{\pi}(a')\hat{\pi}(b)\hat{\pi}(a)|\Psi_0\rangle.\end{aligned} \tag{12}$$

We will always assume that the pointers are prepared in real valued Gaussian states of a width Δf, and $\Delta f'$, respectively. For example, for the first pointer we write

$$G(f) = (2\pi\Delta f^2)^{-1/4}\exp(-f^2/4\Delta f^2), \tag{13}$$

where Δf determines the accuracy (resolution) of the measurement, which is accurate ('strong') when Δf is small, and inaccurate ('weak') when it is large.

The first measurement cannot be affected by the second one (see, for example, [19]), and its readings are distributed according to

$$\rho(f) = \int \mathrm{d}f'\ \rho(f,f') = \int \mathrm{d}a\ G^2(f-a)\langle\Psi_0|\hat{\pi}(a)|\Psi_0\rangle. \tag{14}$$

In general, the second measurement's results are not what they would have been, had the first measurement not been made

$$\begin{aligned}\rho(f') = \int \mathrm{d}f\ \rho(f,f') = \int \mathrm{d}b\ G'^2(f'-b) \\ \times \int \mathrm{d}a\ \mathrm{d}a'\ \exp[-(a-a')^2/8\Delta f^2]\langle\Psi_0|\hat{\pi}(a')\hat{\pi}(b)\hat{\pi}(a)|\Psi_0\rangle,\end{aligned} \tag{15}$$

and would reduce to $\int \mathrm{d}b\ G'^2(f'-b)|\langle\Psi_0|\hat{\pi}(b)|\Psi_0\rangle$ only if the operators $\hat{A}$ and $\hat{B}$ commute, $[\hat{\pi}(a),\ \hat{\pi}(b)] = [\hat{A},\ \hat{B}] = 0$, or if the exponential in the rhs of (15) can be put to unity. This illustrates the well known fact that measurements of quantities such as different components of a spin, or of the particle's position and momentum, must perturb each other. If we wish to recover the classical picture, avoiding this perturbation should be our first priority.

3. Many spins, and as many quantum pointers

Our first attempt at recovering the classical picture, outlined in the Introduction, will involve $K \gg 1$ spin-1/2 systems, all polarised along the z-axis, so the initial state of the whole set is given by the product

$$|\Psi_0\rangle = \prod_{k=1}^{K} |\uparrow z,\ k\rangle. \tag{16}$$

Together, the spins amount to an angular momentum so large (in units of $\hbar$), that we expect it to exhibit certain classical properties. In particular, we should be able to measure its component on any chosen axis to a good accuracy. In addition, successive measurements along various axis should not be affected by their predecessors.

We will also assume that we dispose of K von Neumann pointers, with the positions f_k, $k = 1, \ldots, K$, all prepared in identical Gaussian states of a width Δf, each coupled to one of the spins, and all enacted at the same time, as illustrated in figure 1.

Let the first measurement be along a direction $\vec{n} = (\phi,\ \theta)$, making angles ϕ and θ with the x- and z-axes, respectively. The kth pointer measures (up to a factor of 1/2) the kth spin's component along the chosen axis, the measured operator $\hat{A}_k$ has eigenvalues $A_{1,2} = \pm 1$, and is given by

$$\hat{A}_k = |\uparrow \vec{n},\ k\rangle\langle\uparrow \vec{n},\ k| - |\downarrow \vec{n},\ k\rangle\langle\downarrow \vec{n},\ k| = 2|\uparrow \vec{n},\ k\rangle\langle\uparrow \vec{n},\ k| - 1, \tag{17}$$

where $|\uparrow \vec{n}\rangle$ and $|\downarrow \vec{n}\rangle$ are the spin states aligned up and down a direction $\vec{n}$, respectively and 1 is the identity. In particular, we have

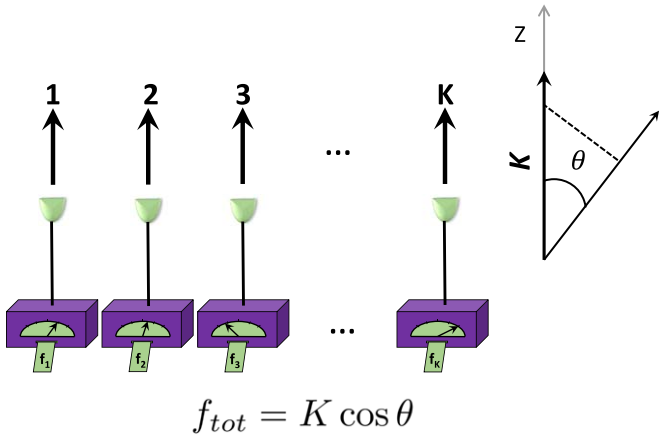

Figure 1. A set of $K \gg 1$ spin-1/2, all polarised along the z-axis, amount to a large angular momentum $K/2$ (in units of $\hbar$), directed along the axis. Coupling quantum pointers, one to each spin, allows one to determine any projection of the total spin to a negligible relative error, and almost without perturbing the spins' state.

$$|\uparrow \vec{n}, k\rangle = \sqrt{w}|\uparrow z, k\rangle + \sqrt{1-w})\exp(i\phi)|\downarrow z, k\rangle,$$
$$|\downarrow \vec{n}, k\rangle = -\sqrt{1-w})|\uparrow z, k\rangle + \sqrt{w}\exp(i\phi)|\downarrow z, k\rangle, \quad (18)$$

where $w \equiv \cos^2(\theta/2)$, and

$$|\uparrow z, k\rangle = \sqrt{w}|\uparrow \vec{n}, k\rangle - \sqrt{1-w})|\downarrow \vec{n}, k\rangle,$$
$$|\downarrow z, k\rangle = [\sqrt{1-w})|\uparrow \vec{n}, k\rangle + \sqrt{w}|\downarrow \vec{n}, k\rangle]\exp(-i\phi). \quad (19)$$

3.1. The CLT

After all metres have fired, we will have a set of $K \gg 1$ pointer readings, $\{f_1, f_2, \ldots, f_K\}$, which we will use to construct a single macroscopic variable

$$f_{\rm tot} = \sum_{k=1}^{K} f_k. \quad (20)$$

Since the pointers are independent, by the CLT [18], the probability to find a reading $f_{\rm tot}$, will tend to a normal distribution

$$\rho(f_{\rm tot})_{K\to\infty} \to (2\pi K\sigma_f^2)^{-1/2}\exp\left[-\frac{(f_{\rm tot} - K\langle f\rangle)^2}{2K\sigma_f^2}\right] \equiv \mathcal{N}(f_{\rm tot}|K\langle f\rangle, K\sigma_f^2), \quad (21)$$

where $\langle f\rangle = \cos\theta$ and $\sigma_f = (\Delta f^2 + \sin^2\theta)^{1/2}$ are the mean and the standard deviation (SD) of each individual measurement, respectively. Thus, the distribution $\rho(f_{\rm tot})$ is centred at K times the average of $\hat{A}$ in the state $|\uparrow z\rangle$, and has a SD $\sqrt{K}$ times larger than it would be for just one of the f_k's. Since the SD grows with K much slower than the largest possible value of the projection, K, we can have a good 'classical' measurement, provided $\sqrt{K}(\Delta f^2 + \sin^2\theta)^{1/2} \ll K$, or if (we remind the reader that $g = \hbar = 1$)

$$\Delta f \ll \sqrt{K}. \quad (22)$$

3.2. The follow up measurement

In order for the classical picture to emerge we need to show that it is possible to choose an accuracy of each individual pointer, Δf, good enough to obtain the expected classical result, $K\cos\theta$, yet poor enough to allow for a subsequent evaluation of the total spin's projection along a different direction, $\vec{n}' = (\phi', \theta')$, yielding the correct result $K\cos\theta'$ with a negligible error.

Let operators

$$\hat{B}_k = |\uparrow \vec{n}', k\rangle\langle\uparrow \vec{n}', k| - |\downarrow \vec{n}', k\rangle\langle\downarrow \vec{n}', k| = 2|\uparrow \vec{n}', k\rangle\langle\uparrow \vec{n}', k| - 1, \quad (23)$$

be measured after the $\hat{A}_k$'s in equation (17) to a new accuracy $\Delta f'$. We need to perturb each spin only slightly, and should choose $\Delta f \gg 1$. Then, expanding the exponentials in equation (15), for the mean reading of each pointer we have

$$\langle f' \rangle \simeq \cos\theta' + O(1/\Delta f^2)$$
$$\sigma^2_{f'} \simeq \Delta f'^2 + \sin^2\theta' + O(\Delta f'^2/\Delta f^2). \quad (24)$$

For the new macroscopic variable, $f'_{\rm tot} \equiv \sum_{k=1}^{K} f'_k$, application of the CLT yields

$$\langle f'_{\rm tot} \rangle \simeq K(\cos\theta' + O(1/\Delta f^2)),$$
$$\sigma^2(f'_{\rm tot}) \simeq K((\Delta f'^2 + \sin^2\theta') + O(\Delta f'^2/\Delta f^2)). \quad (25)$$

Thus the condition

$$1 \ll \Delta f,\ \Delta f' \ll \sqrt{K} \quad (26)$$

allows us to have two good measurements of the total spin components along $\vec{n}$ and $\vec{n}'$, such that $\langle f_{\rm tot} \rangle \simeq K\cos\theta$, $\langle f'_{\rm tot} \rangle \simeq K\cos\theta'$, and $\sigma(f_{\rm tot})/\langle f_{\rm tot} \rangle \simeq \sigma(f'_{\rm tot})/\langle f'_{\rm tot} \rangle \simeq 1/\sqrt{K} \ll 1$. Note that the second measurement will leave the system only slightly perturbed, and ready for the next measurement along some new direction $\vec{n}''$. In this sense the classical picture is recovered.

3.3. An estimate

As an illustration, we evaluate the odds on measuring the x-component, ($\phi = 0, \theta = \pi/2$), of the total spin to a good accuracy, and still find it polarised along the z-axis, ($\phi' = 0,\ \theta' = 0$), if the second measurement is made. Evaluation of the scalar products in equation (15), for the distribution of the second pointer's readings yields

$$\rho(f') = G'^2(f' - 1)(1 + \exp[-1/2\Delta f^2])/2 + G'^2(f' + 1)(1 - \exp[-1/2\Delta f^2])/2. \quad (27)$$

Thus, we have

$$\langle f'_{\rm tot} \rangle = K\exp[-1/2\Delta f^2]$$
$$\sigma^2(f'_{\rm tot}) = K(\Delta f'^2 - \exp[-1/2\Delta f^2] + 1). \quad (28)$$

For a macroscopic sample we expect the number of spins to be of order of the Avogadro number, $K \approx 10^{24}$. Choosing $\Delta f = 10^3$ guarantees that the first measurement would yield $\langle f_{\rm tot} \rangle = 0$, with a SD $\sigma(f_{\rm tot}) \approx \sqrt{2K}\Delta f \sim 10^{15}$, which is 10^9 times smaller than the typical total spin size K. The second measurement will yield a mean value which differs from $K = 10^{24}$ by a factor of the order 10^{18}, or only by about 0.0001% of the measured value. The spread of the readings f' around the mean, not affected by the first measurement, is determined only by the accuracy of the second set of pointers, $\Delta f'$.

3.4. A brief summary

So far, we have described a procedure which transfers the information about the total spin (magnetic moment) of a simple multi-spin system to an ensemble of quantum pointers. The value of the total spin's projection onto an arbitrary axis can then be accurately deduced from the pointers' readings without seriously affecting the state of the spins. Please note certain similarity with the so-called weak measurements [2, 20]. Having achieved the goals (A), (B) and (C) of our wish list in the Introduction, we have made little progress on the (D). It is not clear how an individual pointer could be 'read', as this might require another measuring device to observe the pointer, and yet another device to watch the first device, and so on. Also, the necessity to have 10^{24} individual pointers, is in itself prohibitive. These problems can be remedied, at least to some extent, by considering the so-called collective measurements [10], where the information about a macroscopic property of a system is passed to a single pointer.

4. Collective measurements. Localisation of the wave function

Next we follow [9, 10], and consider a composite of $K \gg 1$ N-dimensional quantum systems, all prepared in the same state $|\psi\rangle$

$$|\Psi_0\rangle = \prod_{k=1}^{K} |\psi_k\rangle. \quad (29)$$

Our aim is to measure the total value of a quantity represented, for each system, by the same operator $\hat{A}$ (see equation (1))

$$\hat{A}_{\rm tot} = \sum_{k=1}^{K} \hat{A}_k, \quad (30)$$

where the subscript k refers to a particular system. The eigenstates of $\hat{A}_{\rm tot}$ are the products $|\underline{n}\rangle \equiv |n_1\rangle|n_2\rangle...|n_K\rangle$, corresponding to the eigenvalues $\sum_{k=1}^{K} A^{n_k}$, $\hat{A}_{\rm tot}|\underline{n}\rangle = \sum_{k=1}^{K} A^{n_k}|\underline{n}\rangle$. Out of N^K eigenvalues of $\hat{A}_{\rm tot}$ only $J \ll N^K$ will be different, and we denote them as $A^{\rm tot}_j$. It is readily seen that $KA_{\rm min} \leqslant A^{\rm tot}_j \leqslant KA_{\rm max}$, where $A_{\rm min}$ and

$A_{\max}$ are the smallest and the largest eigenvalues of $\hat{A}$, respectively. As shown in section 2, the state of the composite after obtaining a reading f is

$$\begin{aligned}\langle f|\Phi_1\rangle &= \sum_{j=1}^{J} G(f-A_j^{\rm tot})\langle\Psi_0|\hat{\pi}_K(j)|\Psi_0\rangle^{1/2}|j\rangle,\\ |j\rangle &\equiv \langle\Psi_0|\hat{\pi}_K(j)|\Psi_0\rangle^{-1/2}\hat{\pi}_K(j)|\Psi_0\rangle\\ \hat{\pi}_K(j) &= \sum_{n_1,\ldots,n_K=1}^{N}|\underline{n}\rangle\,\Delta\left(A_j^{\rm tot}-\sum_{k=1}^{K}A^{n_k}\right)\langle\underline{n}|,\end{aligned} \tag{31}$$

where, as before, $\langle j'|j\rangle = \delta_{jj'}$.

We are interested in the structure of the coefficients multiplying the states $|j\rangle$, and recall that

$$\langle\Psi_0|\hat{\pi}_K(j)|\Psi_0\rangle = \sum_{n_1,\ldots,n_K=1}^{N}\prod_{k=1}^{K}|\langle n_k|\psi_k\rangle|^2\Delta\left(\sum_{k=1}^{K}A^{n_k}-A_j^{\rm tot}\right) \tag{32}$$

is just the probability that the sum of K independent variables equals $A_j^{\rm tot}$. Thus, in the limit $K\gg 1$ the CLT predicts that

$$\langle\Psi_0|\hat{\pi}_K(j)|\Psi_0\rangle^{1/2} \simeq (2\pi K\sigma_A^2)^{-1/4}\exp\left[-\frac{(A_j^{\rm tot}-K\langle A\rangle)^2}{4K\sigma_A^2}\right], \tag{33}$$

where

$$\langle A\rangle \equiv \sum_{n=1}^{N}A^n|\langle n|\psi\rangle|^2,\ \sigma_A^2 \equiv \sum_{n=1}^{N}(A^n)^2|\langle n|\psi\rangle|^2-\langle A\rangle^2 \tag{34}$$

are the mean and variance of the operator $\hat{A}$ in the individual state $|\psi\rangle$. Notably, since the CLT holds for an arbitrary individual distribution, equation (33) is valid for any choice of $\hat{A}$ and $|\psi\rangle$.

The case of an operator with a continuous spectrum can be treated similarly, where we obtain

$$\begin{aligned}\langle f|\Phi_1\rangle &\simeq (2\pi K\sigma_A^2)^{-1/4}\int {\rm d}a\; G(f-a)\exp\left[-\frac{(a-K\langle A\rangle)^2}{4K\sigma_A^2}\right]|a\rangle,\\ |a\rangle &\equiv \langle\Psi_0|\hat{\pi}_K(a)|\Psi_0\rangle^{-1/2}\hat{\pi}_K(a)|\Psi_0\rangle,\\ \hat{\pi}_K(a) &= \int {\rm d}\underline{\nu}\;|\underline{\nu}\;\rangle\,\delta\left(\sum_{k=1}^{K}A^{\nu_k}-a\right)\langle\underline{\nu}|,\end{aligned} \tag{35}$$

with $\int {\rm d}\underline{\nu}\equiv\int\prod_{k=1}^{K}{\rm d}\nu_k$, and $|\underline{\nu}\rangle\equiv\prod_{k=1}^{K}|\nu_k\rangle$. The mean $\langle A\rangle$ and the variance σ_A^2 are still given by equations (34), but with the sums replaced by integrals, $\sum_n\to\int{\rm d}\nu$.

Equations (31)–(35) are our main result so far. With the states $|j\rangle$ and $|a\rangle$ appropriately normalised, we can say that the wave function is 'localised' in the region of width $\sim\sqrt{K}$ around a 'macroscopic' value $K\langle A\rangle$, with negligible contributions from the $|j\rangle$ and $|a\rangle$ outside this region. Thus, it is possible to choose a measurement's accuracy Δf small enough for the error, relative to the typical large value of the measured quantity, to be small, $\Delta f\ll K(A_{\max}-A_{\min})$. At the same time, it is possible to have Δf large enough for $G(f)$ to be practically constant for all important states in the decompositions (31) or (35), $\Delta f\gg\sqrt{K}\,\sigma_A$. With the state of the composite system barely changed, the system is ready for the next collective measurement, not affected by the previous ones. In summary, we can have a good (accurate) classical measurement, provided

$$\sqrt{K}\,\sigma_A \ll \Delta f \ll K(A_{\max}-A_{\min}). \tag{36}$$

The measurement is 'classical', firstly, because its single realisation yields the classical value with a negligible relative error and, secondly, because its back action on the measured system is negligible as well. The localisation property of a wave function, describing a large conglomerate of non-interacting components, must hold in every representation, and for all additive quantities. Our ability to assign to a macroscopic system in a quantum state a set of 'objectively existing' classical values signals, therefore, return to the classical picture.

5. Many spins, and only one quantum pointer

Next we apply the approach of the previous section to a measurement of the component of the total spin along a direction $\vec{n}$, for a system in the state (16), see figure 2. The corresponding operator (up to a factor of 1/2) is given by

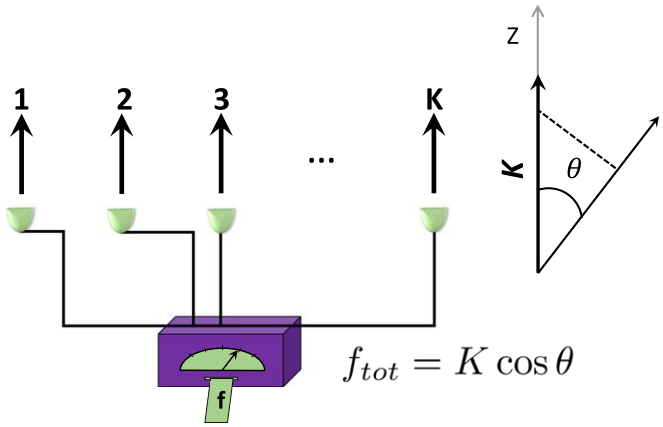

Figure 2. A collective measurement. Coupling a single quantum pointer to each one of the K spins also allows one to determine any projection of the total angular momentum to a negligible relative error while leaving the state of the spins virtually intact.

$$\hat{A}_{\rm tot} = \sum_{k=1}^{K} \hat{A}_k = 2\sum_{k=1}^{K} |\uparrow \vec{n},\, k\rangle\langle \uparrow \vec{n},\, k| - 1. \tag{37}$$

There are $K+1$ eigenvalues $A_j^{\rm tot} = 2j - K, j = 0, 1, \ldots, K$. $\hat{\pi}_K(j)$ projects onto a subspace spanned by all possible products of the states $|\uparrow \vec{n}\rangle$ and $|\downarrow \vec{n}\rangle$, containing precisely j states $|\uparrow \vec{n}\rangle$. There are $C_j^K = K!/j!(N-j)!$ such products, $\langle \uparrow z|\uparrow \vec{n}\rangle = \cos(\theta/2)$, and $\langle \uparrow z|\downarrow \vec{n}\rangle = \sin(\theta/2)$. Using well known properties of the binomial distribution [21], we obtain for $K \gg 1$

$$\langle\Psi_0|\hat{\pi}_K(j)|\Psi_0\rangle = C_j^K[\cos(\theta/2)]^{2j}\,[\sin(\theta/2)]^{2(K-j)} \overset{K\to\infty}{\longrightarrow} \mathcal{N}[\,j|K\cos^2(\theta/2),\, K\sin^2(\theta)/4]. \tag{38}$$

Since $j = (A_j^{\rm tot} + K)/2$ it follows that

$$\langle f|\Phi_1\rangle \simeq (2\pi K\sin^2\theta)^{-1/4}\sum_{j=0}^{K} G(f - A_j^{\rm tot})\exp\left[-\frac{(A_j^{\rm tot} - K\cos\theta)^2}{4K\sin^2\theta}\right]|j\rangle, \tag{39}$$

where $\langle j'|j\rangle = \delta_{jj'}$, see figure 3. Note that equation (39) can be obtained directly from equations (31)–(33), by noting that

$$\langle A\rangle = \langle \uparrow z|\hat{A}|\uparrow z\rangle = \cos\theta,\quad \sigma_A = \sqrt{1 - \langle A\rangle^2} = \sin\theta. \tag{40}$$

For a Gaussian pointer (13), evaluation of Gaussian integrals yields for the distribution of the readings

$$\rho(f) = [2\pi(\Delta f^2 + K\sin^2\theta)]^{-1/2}\ \exp\left[-\frac{(f - K\cos\theta)^2}{2(\Delta f^2 + K\sin^2\theta)}\right]. \tag{41}$$

Thus, for the number of spins sufficiently large, there are many possibilities to realise a 'classical' measurement, as described in the previous section. Indeed, with $\Delta f \sim K^{(1+\epsilon)/2}, 0 < \epsilon < 1$ we achieve, as $K \to \infty$,

$$\sqrt{2K}|\sin\theta| \ll \Delta f \ll A_{\rm max} = K. \tag{42}$$

More precisely, if one writes $K = 10^n$ then n and ϵ satisfy

$$n + \alpha \leqslant n + \epsilon \leqslant 2n,\quad (\alpha = \log_{10} 2). \tag{43}$$

For instance, considering a number $K = 10^8$ $(n = 8)$ of particles, one can set $\epsilon = (\alpha + n)/2n$, that yields $\Delta f \sim 10^6$ and $\sqrt{2K} \sim 10^4$. Hence, all quantities involved in equation (42) differ by two order of magnitude.

5.1. Damage to the initial state

It is easy to assess the damage done to the state $|\Psi_0\rangle$, provided the pointer reads f. A convenient measure of the change produced by the measurement is the norm of the difference between $|\Psi_0\rangle$ and the properly normalised final state $\langle f|\Phi_1\rangle/\sqrt{\rho(f)}$ (recall that $\rho(f) = \langle f|\Phi_1\rangle\langle\Phi_1|f\rangle$),

$$\mathrm{Err}(f) \equiv \left(\sum_{j=0}^{K}\langle\Psi_0|\hat{\pi}_K(j)|\Psi_0\rangle\left|1 - \frac{G(f - A_j^{\rm tot})}{\sqrt{\rho(f)}}\right|^2\right)^{1/2}. \tag{44}$$

Replacing the sum over j by an integral $\int_{-\infty}^{\infty} {\rm d}A^{\rm tot}$, evaluating several Gaussian integrals, and taking the limit $\Delta f \gg \sqrt{2K}\sin\theta$, we find

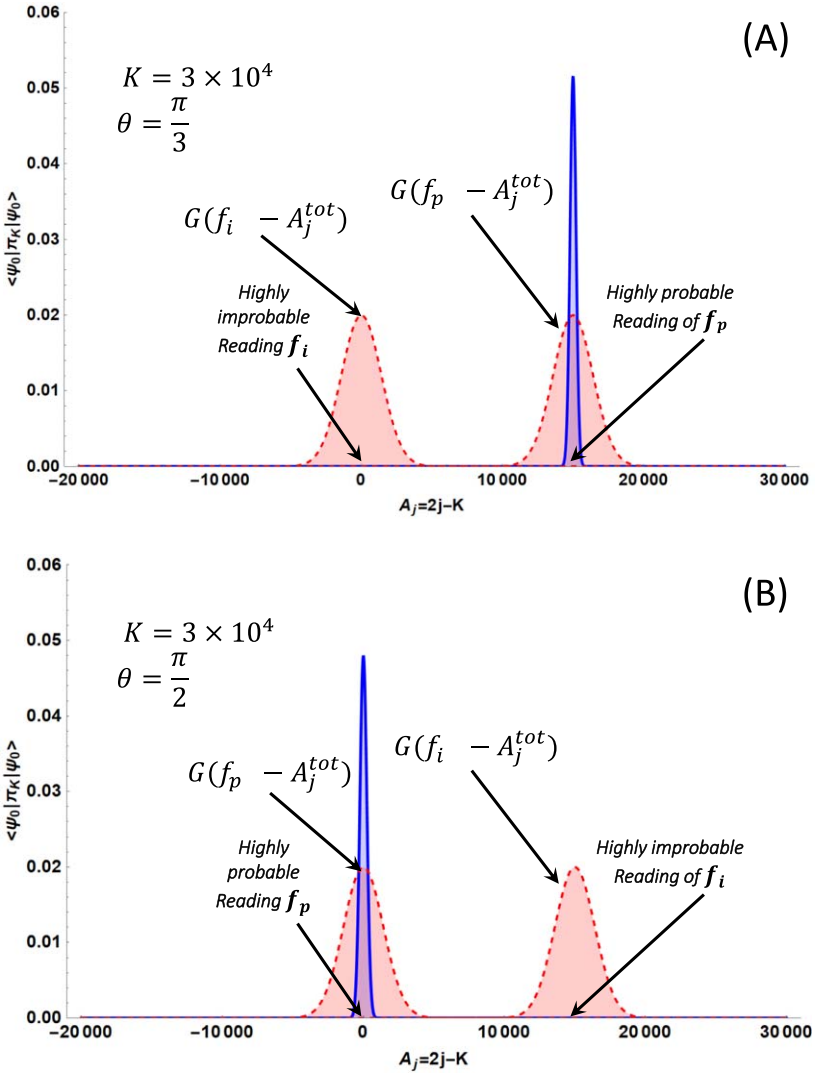

Figure 3. (A) Localisation of the wave function. Coefficients, multiplying the states $|j\rangle$ in equation (3) in the expansion of the state (16) for $K = 3 \times 10^4$ spins, all polarised along the z-axis. a) If the projection at $\theta = \pi/3$ is measured. Also shown (dashed) is the function $G(f - A_j^{\rm tot})$. A reading f is probable if $G(f - A_j^{\rm tot})$ overlaps with the region of support of the wave function. (B) The same than figure (A) for $\theta = \pi/2$.

$$\rho(f) \simeq (2\pi\Delta f^2)^{-1/2} \exp\left[-\frac{(f - K\cos\theta)^2}{2\Delta f^2}\right],$$
$$\mathrm{Err}(f) = \sqrt{2}\left(1 - \exp\left[-\frac{(f - K\cos\theta)^2}{(4\Delta f^2/\sqrt{2K}\sin\theta)^2}\right]\right)^{1/2}, \quad (45)$$

with the second Gaussian in equations (45) much broader than the first one, and $\mathrm{Err}(f)$ will stay close to zero for all readings f, which are likely to occur in the measurement. Thus,

$$\Delta f \sim K^{(1/2+\epsilon)}, \quad 0 < \epsilon < 1/2, \quad (46)$$

would be a suitable choice, provided the number of spins, K, is sufficiently large. Indeed, we obtain $\mathrm{Err}(f_r) = r < 1$, provided the pointer reads $f_r = \pm(4\Delta f^2/\sqrt{2K}\sin\theta)|\ln(1 - r^2/2)|^{1/2} + K\cos\theta$. Now the probability to have an error greater than r is (adding a factor of 2 for the two tails of the Gaussian)

$$\mathrm{Prob}(\mathrm{Err} \geqslant r) = 2\int_{f_r}^{\infty} \rho(f)\mathrm{d}f = \mathrm{erfc}\left(\frac{f_r - K\cos\theta}{\sqrt{2}\,\Delta f}\right) \simeq \frac{\sqrt{K}\sin\theta}{2\sqrt{\pi}\Delta f|\ln(1 - r^2/2)|^{1/2}}\left(1 - \frac{r^2}{2}\right)^{\frac{4\Delta f^2}{K\sin^2\theta}}, \quad (47)$$

which duly tends to zero as $\Delta f/\sqrt{K}\sin\theta \to \infty$.

5.2. The follow up measurement

Next we want to look for a regime in which a measurement of the total spin's component

$$\hat{B}_{\rm tot} = \sum_{k=1}^{K} [|\uparrow \vec{n}', k\rangle\langle\uparrow \vec{n}', k| - |\downarrow \vec{n}', k\rangle\langle\downarrow \vec{n}', k|], \tag{48}$$

on a different direction $\vec{n}' = (\phi', \theta')$ will not be affected by previously measuring it along $\vec{n} = (\phi, \theta)$. The corresponding distribution $\rho(f')$ is now given by a discrete sum (see equation (15))

$$\rho(f') = \sum_{j=0}^{K} G^2(f' - B_j^{\rm tot}) \sum_{m,m'=0}^{K} \langle\Psi_0|\hat{\pi}_K(m')\hat{\pi}_K(j)\hat{\pi}_K(m)|\Psi_0\rangle \times \exp\left[-\frac{(A_m^{\rm tot} - A_{m'}^{\rm tot})^2}{8\Delta f^2}\right], \tag{49}$$

where $A_m^{\rm tot} = 2m - K$, $B_j^{\rm tot} = 2j - K$, and the projectors $\hat{\pi}_K(m)$, $\hat{\pi}_K(m')$ and $\hat{\pi}_K(j)$, for $\hat{A}_{\rm tot}$ and $\hat{B}_{\rm tot}$, respectively, are defined in equations (31). Calculation of $\rho(f')$ with the help of equation (15) would require evaluation of numerous scalar products, and we will ask a simpler question instead. As in section 3.3, we will try to measure the z-component of the total spin, with and without measuring of $\hat{A}_{\rm tot}$ first. With no measurement of $\hat{A}_{\rm tot}$ made, only one Gaussian will be present in the sum (49)

$$\rho(f') = G^2(f' - K), \tag{50}$$

since $|\Psi_0\rangle$ is the eigenstate of $\hat{B}_{\rm tot}$, $\hat{B}_{\rm tot}|\Psi_0\rangle = K|\Psi_0\rangle$. With the measurement of $\hat{A}_{tot}$ made, for the coefficient multiplying $G^2(f' - K)$ in the sum (15) we have

$$\begin{aligned}
&\sum_{j,j'=0}^{K} \langle\Psi_0|\hat{\pi}_K(j')|\Psi_0\rangle\langle\Psi_0|\hat{\pi}_K(j)|\Psi_0\rangle \exp\left[-(A_j^{\rm tot} - A_{j'}^{\rm tot})^2/8\Delta f^2\right] \\
&\approx \int \frac{{\rm d}A{\rm d}A'}{2\pi K\sin^2\theta} \exp\left[-\frac{(A - K\cos\theta)^2}{2K\sin^2\theta} - \frac{(A' - K\cos\theta)^2}{2K\sin^2\theta} - \frac{(A - A')^2}{8\Delta f^2}\right] \\
&= (1 + K\sin^2\theta/2\Delta f^2)^{-1/2} \simeq 1 - \frac{K\sin^2\theta}{4\Delta f^2},
\end{aligned} \tag{51}$$

which tends to unity for $\Delta f \gg \sqrt{K}$. We, therefore, have a condition for a good classical measurement of any projection of the system's total spin (see equations (42))

$$\sqrt{K} \ll \Delta f \ll K, \tag{52}$$

easily satisfied for $K \gg 1$.

5.3. An estimate

To make our arguments plausible, we need to check whether this section's simple model is at least in the right ballpark. The Avogadro constant, $N_A \approx 6 \times 10^{23}$ is a reasonable estimate for the number of constituent parts of a 'macroscopic' object, and we will use it throughout the rest of the paper.

In the previous example, for $K \sim 10^{24}$, the maximum size of the total spin (in units of $\hbar$) is of order of 10^{24}. For the 'characteristic size' of the spin state from (39) we have

$$\sqrt{2K}\sin\theta \sim 10^{12}. \tag{53}$$

Choosing Δf a thousand times larger, $\Delta f \sim 10^{15}$, guarantees a good measurement of the x-component (see equations (42))

$$\langle f\rangle = 0, \quad \sigma(f) \simeq \Delta f \sim 10^{15} \ll K. \tag{54}$$

The measurement is not likely to change the state of the spins. From equation (47) the probability to incur an error of more than 1% is negligible. Finally, equation (51) shows that the coefficient multiplying $G^2(f' - K)$ in the expression for the distribution of the second pointer's readings (15), is very close to unity. With all coefficients multiplying the G's in the sum (15) non-negative, and all $|B_j^{\rm tot}| \leqslant K$ we find the mean reading of the second pointer close to its unperturbed value

$$\langle f'\rangle = K + \delta\langle f'\rangle, \quad |\delta\langle f'\rangle| \leqslant 2 \times 10^{-6}K \ll K. \tag{55}$$

We have, therefore, two good non-perturbing 'classical' measurements along two non-collinear axes, which leave the system of the chosen size ready for more measurements of this type.

6. Many quantum particles, and only one quantum pointer

Our next example involves $K \gg 1$ non-interacting free particles, all in the same quantum state $|\psi\rangle$, with a mean momentum p_0,

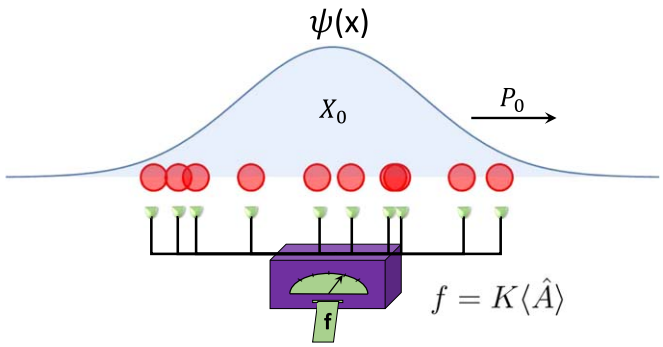

Figure 4. Collective measurement on a cloud of $K \gg 1$ free particles, all in the same state $|\psi\rangle$. Coupling a single quantum pointer to all of the particles also allows one to determine the value of any additive quantity $\hat{A}$ to a negligible relative error, while leaving the state of the spins virtually intact. The cases of $\hat{A}$ representing the coordinate and the momentum are analysed in section 7.

$$|\Psi_0\rangle = \prod_{k=1}^{K} |\psi_k\rangle, \quad |\psi_k\rangle = \int \psi(x_k)|x_k\rangle \, \mathrm{d}x_k. \tag{56}$$

We will be interested in determining the position of the system's COM, as well as its total momentum, see figure 4. Thus, we consider operators

$$\hat{X} = \sum_{k=1}^{K} \int \mathrm{d}x_k |x_k\rangle \, x_k \, \langle x_k| \equiv \sum_{k=1}^{K} \hat{x}_k, \tag{57}$$

such that $\hat{X}_{\mathrm{COM}} = \hat{X}/K$, and

$$\hat{P} = \sum_{k=1}^{K} \int \mathrm{d}p_k |p_k\rangle \, p_k \, \langle p_k| \equiv \sum_{k=1}^{K} \hat{p}_k |p_k\rangle \equiv (2\pi)^{-1/2} \int \mathrm{d}x_k \exp[\mathrm{i}p_k x_k] |x_k\rangle, \tag{58}$$

as well as two Gaussian pointers, with positions f and f'.

6.1. Position of the COM

With the help of equation (35), we find the corresponding wave function to be

$$\langle f|\Phi_1\rangle \approx (2\pi K\sigma_x^2)^{-1/4} \int_{-\infty}^{\infty} \mathrm{d}a \; G(f-a) \exp\left[-\frac{(a - K\langle x\rangle)^2}{4K\sigma_x^2}\right] |a\rangle, \tag{59}$$

where

$$\langle x\rangle = \int \mathrm{d}x \; x \, |\psi(x)|^2, \quad \sigma_x^2 = \int \mathrm{d}x \; x^2 \, |\psi(x)|^2 - \langle x\rangle^2. \tag{60}$$

The precise form of the states $|a\rangle$, $\langle a|a'\rangle = \delta(a - a')$, defined in equation (35), is of no importance to us. It is sufficient to note that the wave function (59) is localised in a region of a size $\Delta a = 2\sqrt{K}\sigma_x$ around $a = K\langle x\rangle$, and that a measurement to an accuracy $\Delta f \gg \sqrt{K}$ will yield $\langle f\rangle = K\langle x\rangle$, and leave the state of all K particles virtually unchanged. For example, for the particles in identical Gaussian states with a mean momentum p_0,

$$\psi(x_k) = (2\pi\sigma_x^2)^{-1/4} \exp\left(-(x_k - x_0)^2/4\sigma_x^2 + \mathrm{i}p_0 x_k\right), \tag{61}$$

for the meter readings we find

$$\rho(f) = \int \mathrm{d}\underline{x} \; G^2\left(f - \sum_k x_k\right) |\langle \underline{x}|\Psi_0\rangle|^2 = (2\pi(\Delta f^2 + K\sigma_x^2))^{-1/2} \exp\left[-\frac{(f - Kx_0)^2}{2(\Delta f^2 + K\sigma_x^2)}\right], \tag{62}$$

See figure 5. To describe the COM of the cloud of particles, we need to rescale f by a factor of K, thus introducing $f_{\mathrm{COM}} = f/K$, $\Delta f_{\mathrm{COM}} = \Delta f/K$, distributed as

$$\rho(f_{\mathrm{COM}}) = [2\pi(\Delta f_{\mathrm{COM}}^2 + \sigma_x^2/K)]^{-1/2} \exp\left[-\frac{(f_{\mathrm{COM}} - x_0)^2}{2(\Delta f_{\mathrm{COM}}^2 + \sigma_x^2/K)}\right], \tag{63}$$

which for $K \gg \Delta f \gg \sqrt{K}$ tends to a Gaussian distribution with a SD $\sim \Delta f/K$, so that $\rho(f_{\mathrm{COM}}) \to \delta(f_{\mathrm{COM}} - x_0)$ as $K \to \infty$. We can also evaluate the final mixed state of the particles, $\hat{R}_{\mathrm{part}}$

$$\langle a'|\hat{R}_{\rm part}|a\rangle = \langle \Phi_1|a'\rangle\langle a|\Phi_1\rangle = \exp\left[-\frac{(a-a')^2}{8\Delta f^2}\right]\langle a'|\Psi_0\rangle\langle\Psi_0|a\rangle. \tag{64}$$

The terms $\langle a'|\Psi_0\rangle$ and $\langle\Psi_0|a\rangle$ in this last equation are just the pure state of the particles before the measurement. By equation (59), the difference $(a-a')^2$ is of order of $K\sigma_x^2$. Thus, for

$$\Delta f_{\rm COM} \gg \sigma_x/\sqrt{K}, \tag{65}$$

the measurement yields the position of the COM of the composite system to an accuracy $\Delta f_{\rm COM}$, without seriously affecting its quantum state. Next we want to see whether such a measurement would still allow us to accurately determine the cloud's momentum as well.

6.2. Follow up measurement of the total momentum

Rather than evaluate the damage to the cloud's state which the measurement of the COM is likely to produce, we will go straight to the distribution of the second pointer's reading. From (15) we have

$$\rho(f') = (2\pi)^{-K}(2\pi\sigma_x^2)^{-K/2}\int {\rm d}\underline{p}\,G'^2(f'-\sum_k p_k)\times\int {\rm d}\underline{x}\int {\rm d}\underline{x}'$$
$$\times\exp\left[-\frac{\left(\sum_k(x_k-x_k')\right)^2}{8\Delta f^2}\right]\exp\left[\sum_k\left(-\frac{(x_k^2+x_k'^2)}{4\sigma_x^2}+{\rm i}(p_0-p_k)(x_k-x_k')\right)\right]. \tag{66}$$

Evaluating the Gaussian integrals, after some algebra we obtain

$$\rho(f') = (\pi\delta f'^2)^{-1/2}\exp\left[-\frac{(f'-Kp_0)^2}{\delta f'^2}\right], \tag{67}$$

where

$$\delta f' = \left(2\Delta f'^2+\frac{K}{2\sigma_x^2}+\frac{K^2}{2\Delta f^2}\right)^{1/2}. \tag{68}$$

Thus, on average, the pointer points towards the correct value of the total momentum of the cloud, $\langle f'\rangle = Kp_0$. As was expected, choosing the accuracy of the first measurement in such a way that $\sqrt{K}\ll\Delta f\ll K$ guarantees the it does not affect the measurement of the momentum, since $\delta f'$ tends to its unperturbed value, $\delta f'\approx(2\Delta f'^2+K/2\sigma_x^2)^{1/2}$. Finally, choosing $\sqrt{K}\ll\Delta f'\ll K$ provides for a good classical measurement of the momentum, which leaves the state of the cloud practically unperturbed, and ready for the next classical observation.

6.3. The classical trajectory

Since the first measurement of the position of the COM (more precisely, of the operator $\hat{X}$, see equation (57)) at $t=0$ appears to perturb the state of the cloud only slightly, we should be able to make a second measurement at some $t>0$, and find the COM where the classical mechanics would put it, in our case, displaced by $vt=p_0t/m$, m being the particle's mass. Taking $x_0=0$ in equation (61), from (15) we have

$$\rho(f') = \int {\rm d}\underline{y}\;G'^2\left(f'-\sum_k y_k\right)\times\int {\rm d}\underline{x}\;{\rm d}\underline{x}'\exp\left[-\frac{\left(\sum_k(x_k-x_k')\right)^2}{8\Delta f^2}\right]$$
$$\times\langle\Psi_0|\underline{x}'\rangle\langle\underline{x}'|\hat{U}^{-1}(t)|\underline{y}\rangle\langle\underline{y}|\hat{U}(t)|\underline{x}\rangle\langle\underline{x}|\Psi_0\rangle, \tag{69}$$

where

$$\langle\underline{y}|\hat{U}(t)|\underline{x}\rangle = \left(\frac{2\pi{\rm i}t}{m}\right)^{-K/2}\prod_k\exp\left[\frac{{\rm i}m(x_k-y_k)^2}{2t}\right] \tag{70}$$

is the propagator for the cloud of non-interacting particles. After evaluation of the Gaussian integrals involved we obtain

$$\rho(f'_{\rm COM}) = (\pi\delta f'^2_{\rm COM})^{-1/2}\exp\left[-\frac{(f'_{\rm COM}-vt)^2}{\delta f'^2_{\rm COM}}\right], \tag{71}$$

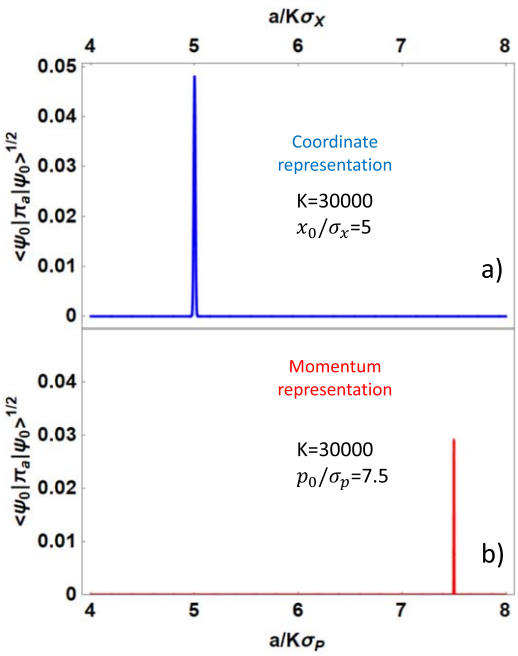

Figure 5. Localisation of the wave function around the classical values. Coefficients, multiplying the states $|a\rangle$ in the expansion (9), for $K = 3 \times 10^4$ free particles, all in the same Gaussian state $|\psi\rangle$ (61), with $x_0/\sigma_x = 5.0$, and $p_0/\sigma_p = p_0\sigma_x/2 = 7.5$. (a) If the position of the centre of mass is measured; (b) for the measurement of the total momentum.

where

$$\delta f'^2_{\rm COM} = \frac{2\Delta f'^2}{K^2} + K^{-1}\left(2\sigma_x^2 + \frac{t^2}{2m^2\sigma_x^2}\right) + \frac{t^2}{2m^2\Delta f^2}. \tag{72}$$

We note that we are in the 'classical regime', provided the first two terms in the rhs of equation (72) are dominant. As expected, for a given time t, the effect of the first measurement disappears for $\Delta f \gg \sqrt{K}$, and the second measurement becomes a 'good' classical one, for $\sqrt{K} \ll \Delta f' \ll K$. Remaining within these limits, one will always stay in the classical regime, where measurements describe a large quantum cloud of particles as a classical point-sized object, which follows a well defined trajectory.

7. Splitting a cloud of particles by scattering

To extend the discussion beyond free motion, we assume that our cloud of particles, travelling from left to right, meets with a potential barrier which is non-zero only between $x = -d$ and $x = 0$, see figure 6. If one waits long enough, each particle's state will be split into the transmitted (T) and reflected (R) parts

$$|\psi_k\rangle = |\psi_k^T\rangle + |\psi_k^R\rangle, \tag{73}$$

localised far to the right and to the left of the barrier, respectively. Now the COM of the system lies somewhere between $|\psi^T\rangle$ and $|\psi^R\rangle$, where no particles are found, and its position is of little interest. We can, however, specify to the transmission channel, by considering two commuting operators ($\Theta(x) = 1$ for $x > 0$, and 0 otherwise)

$$\begin{aligned}\hat{N}^T &= \sum_{k=1}^{K}\int |x_k\rangle\,\Theta(x_k)\,\langle x_k|{\rm d}x_k \equiv \sum_{k=1}^{K}\hat{n}_k^T \\ \hat{X}^T &= \sum_{k=1}^{K}\int |x_k\rangle\,\Theta(x_k)\,\langle x_k|\,x_k\,{\rm d}x_k \equiv \sum_{k=1}^{K}\hat{x}_k^T,\end{aligned} \tag{74}$$

of which the first represents the number of the transmitted (T) particles, N^T, and the second is related to the position of the COM of the transmitted cloud as $X^T_{\rm COM} = X^T/N^T$. With $|\psi_k\rangle$ split into only two orthogonal components, the analysis is similar to that of the spin-1/2 case of section 6. The averages and variances of one-particle operators are

$$\langle\hat{n}^T\rangle = \langle\psi^T|\psi^T\rangle \equiv P^T, \quad \sigma^2_{n^T} = P^T(1 - P^T), \tag{75}$$

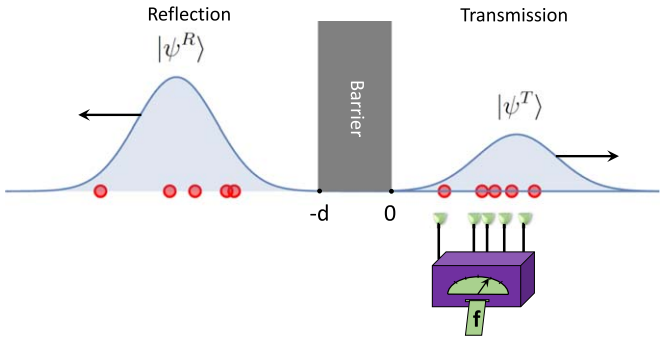

Figure 6. A cloud of free particles, scattered off a potential barrier, is split into the transmitted (T), and reflected (R) parts. A quantum pointer, coupled only to the particles, found to the right of the barrier, determines the position of the centre of mass of the transmitted cloud.

if $\hat{N}^T$ is measured, and

$$\langle x^T\rangle = \langle\psi^T|x|\psi^T\rangle,$$
$$\sigma^2_{x^T} = \langle\psi^T|x^2|\psi^T\rangle - \langle\psi^T|x|\psi^T\rangle^2, \tag{76}$$

for a measurement of $\hat{X}^T$. Denoting the corresponding pointer readings as f_{N^T} and f_{X^T}, from equation (35) we obtain

$$\langle f_{N^T}|\Phi_1\rangle \simeq (2\pi K\sigma^2_{n^T})^{-1/4} \times \int da\; G(f_{N^T} - a)\exp\left[-\frac{(a - KP^T)^2}{4K\sigma^2_{n^T}}\right]|a\rangle \tag{77}$$

and

$$\langle f_{X^T}|\Phi_1\rangle \simeq (2\pi K\sigma^2_{x^T})^{-1/4} \times \int da\; G(f_{X^T} - a)\exp\left[-\frac{(a - K\langle x^T\rangle)^2}{4K\sigma^2_{x^T}}\right]|a\rangle, \tag{78}$$

where the states $|a\rangle$ in equations (77) and (78) are defined as in equation (35) for the operators $\hat{N}^T$ and $\hat{X}^T$, respectively. Thus, with

$$\sqrt{KP^T(1 - P^T)} \ll \Delta f_{X^T} \ll PK, \quad \text{and} \quad \sigma_{x^T} \ll \Delta f_{N^T} \ll K\langle x^T\rangle, \tag{79}$$

it is possible to have good classical measurements of both the number of transmitted particles, and the position of the COM of the transmitted cloud, for $K \gg 1$. The same can be repeated for the reflected part of the cloud, by replacing $\Theta(x_k)$ by $\Theta(-x_k - d)$ in equations (74). Such measurements, performed before and after the particles interact with the barrier, yield a picture of a point sized object of a mass mK being divided into two parts of masses $P^T mK$ and $(1 - P^T)mK$, moving to the right and to the left, respectively.

8. Quantum measurements with an observer

Now we can consider an Observer, existing in hugely oversimplified world of objects, made up from small parts which obey quantum mechanical rules. Some of the objects are large, due to the large number of their constituent parts. For reasons unknown to us, the Observer can only gain information about the collective properties of objects, such as the total spin, position of the COM, or the total momentum, by applying a single-pointer measurement procedure, described above. The accuracy of the measurements is always good enough to ensure an error small relative to the large value measured, yet sufficiently poor so as not to perturb the quantum state of a sufficiently large conglomerate. We must conclude that such an Observer, dealing with large objects, would perceive an essentially classical world in which all components of magnetic moments can be measured simultaneously. He/she/it would also visualise a cloud of quantum particles as a single small object, possessing a well defined position and momentum at all times, and moving along a classical trajectory, prescribed by classical mechanics.

This suggests using the COM of the cloud as a pointer. It was shown in section 7, that the COM's position can be determined to a sufficient accuracy, without significantly altering its quantum state. Thus the result of a measurement, encoded in its position, can be verified by other independent Observers, thus becoming, in Einstein's words [22], an 'element of reality'.

There remains one delicate question, namely how exactly would an Observer observe the pointers, which provide for his/her/its information about the outside world? Here we will need to make a strong assumption, endowing the Observer with an ability to simply 'see' the large objects (as their COM's) in the coordinate space, in the way one is able to read an analogue car's speedometer without the help of additional intermediary device. Thus, in what follows, the coordinate space will have to have a special status, in the sense that the total spin, or the total angular momentum could not be 'seen' directly, but the COM of a cloud of particles could. We will go one step further and equip the Observer with a sensor, a quantum pointer of a suitable resolution Δf, and identify the Observer's state with the state of the pointer, prescribed by the conventional quantum mechanical rules. We will say nothing about Observer's conscience, its status, or it adherence to quantum, or any other laws. Admittedly, the above is a less than perfect model for the immensely more complex physical world. However, we only wish to prove a principle, and will use it throughout the rest of the paper.

9. One quantum spin, one macroscopic pointer, and one or more observers

Having recovered certain degree of classicality for a composite, consisting of many quantum particles, we can now devise a measurement, which can make properties of a quantum system, such as a spin-1/2, directly accessible to our rudimentary Observer (see figure 7). Now the pointer itself will be a cloud of $K \gg 1$ particles in the same Gaussian state

$$|\psi_k\rangle = \int \psi(x_k)|x_k\rangle\, \mathrm{d}x_k, \quad k = 1, 2,\ldots,K$$
$$\psi(x_k) = (2\pi\sigma_x^2)^{-1/4}\exp[-x_k^2/4\sigma_x^2], \tag{80}$$

each coupled to a single spin in a state

$$|\varphi_0\rangle = \alpha|\uparrow z\rangle + \beta|\downarrow z\rangle, \tag{81}$$

so that the full interaction Hamiltonian is given by

$$\hat{H}_{\rm int} = -\mathrm{i}g\delta(t)\sum_{n=1}^{N}\partial_{x_n}[|\uparrow z\rangle\langle\uparrow z| - |\downarrow z\rangle\langle\downarrow z|], \tag{82}$$

where the coupling strength g was reinstated for further convenience. (Note that since $\partial_X = K\sum_{i=1}^{K}\partial_{x_i}$, this is equivalent to coupling the spin to the COM of the cloud, albeit with a much smaller strength, g/K.)

Thus, our purpose is to measure (up to a factor 1/2) the z-component of the spin, using the COM of the cloud of quantum particles as a 'classical' pointer. With the states of all particles translated by either g or $-g$, for the state of the composite spin+particles+pointer (the last one represents the Observer, as discussed in the previous section) we have

$$\langle f|\Phi_1\rangle = \alpha\int \mathrm{d}\underline{x}\; G\left(f - \sum_k x_k\right)|\Psi_{\rm part}(\underline{x}, +g)\rangle|\uparrow z\rangle + \beta\int \mathrm{d}\underline{x}\; G\left(f - \sum_k x_k\right)|\Psi_{\rm part}(\underline{x}, -g)\rangle|\downarrow z\rangle, \tag{83}$$

where

$$|\Psi_{\rm part}(\underline{x}, \pm g)\rangle \equiv \prod_{k=1}^{K}\psi(x_k \mp g)|x_k\rangle \tag{84}$$

is the state of the cloud, shifted as a whole by g to the right, or to the left, respectively. As before, the probability to have a reading $f_{\rm COM} = f/K$, which is all the Observer can 'see', is given by tracing out the spin's and particles's variables from the pure state $|\Phi_1\rangle\langle\Phi_1|$, and, recalling the derivation of equation (62), we find

$$\rho(f_{\rm COM}) = [2\pi(\Delta f_{\rm COM}^2 + \sigma_x^2/K)]^{-1/2} \times\left(|\alpha|^2\exp\left[-\frac{(f_{\rm COM} - g)^2}{2(\Delta f_{\rm COM}^2 + \sigma_x^2/K)}\right] + |\beta|^2\exp\left[-\frac{(f_{\rm COM} + g)^2}{2(\Delta f_{\rm COM}^2 + \sigma_x^2/K)}\right]\right). \tag{85}$$

If the Observer's own accuracy, Δf, is such that

$$\sigma_x^2/K \ll \Delta f_{\rm COM} = \Delta f/K \ll g, \tag{86}$$

the two narrow Gaussians in equation (85) do not overlap, and represent a binary choice of finding the spin aligned up or down the z-axis, the chances of that being $|\alpha|^2$ and $|\beta|^2$, respectively.

At the same time, $G(f - \sum_k x_k)$ is broad enough to leave the highly localised (see equation (35)) states of the particles almost unchanged. Approximating $G(f - \sum_k x_k)$ by $G(f)$ in equation (83), for the mixed state of the particles of the macroscopic pointer we obtain (see equation (64))

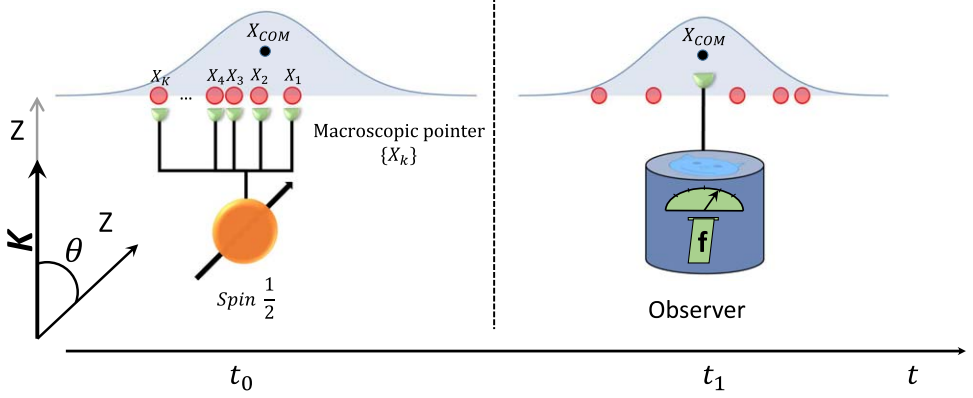

Figure 7. A macroscopic pointer, consisting of $K \gg 1$ free particles in the same state, and coupled to a single spin-1/2, accurately measures at time t_1 the spin's projection on the z-axis, possibly destroying the state of the spin. Later, at time t_2, an Observer determines the position of the pointer's centre of mass, causing only negligible damage to the state of the pointer (see figure 4).

$$\hat{R}_{\rm part} \simeq |\alpha|^2 |\Psi_{\rm part}(+g)\rangle \langle \Psi_{\rm part}(+g)| + |\beta|^2 |\Psi_{\rm part}(-g)\rangle \langle \Psi_{\rm part}(-g)|, \tag{87}$$

where $|\Psi_{\rm part}(\pm g)\rangle = \int {\rm d}\underline{x}\ |\Psi_{\rm part}(\underline{x}, \pm g)\rangle$.

Finally, this accurate (or 'strong') measurement results in the destruction of the spin's state, whose density matrix becomes diagonal,

$$\hat{\mathcal{R}}_{\rm spin} \simeq |\alpha|^2 |\uparrow z\rangle \langle \uparrow z| + |\beta|^2 |\downarrow z\rangle \langle \downarrow z|, \tag{88}$$

since the coherences rapidly vanish as the number of particles, K, increases,

$$\langle \downarrow z|\hat{\mathcal{R}}_{\rm spin}|\uparrow z\rangle = \langle \uparrow z|\hat{\mathcal{R}}_{\rm spin}|\downarrow z\rangle^* \sim \alpha\beta^* \exp(-Kg/2\sigma_x^2) \to 0. \tag{89}$$

These results can be presented in a slightly different manner. After the pointer had interacted with the spin, but before the Observer 'looked' at it, the entangled state of the spin+pointer subsystem $|\Phi_1'\rangle$ is given by

$$|\Phi_1'\rangle = \alpha|\Psi_{\rm part}(+g)\rangle|\uparrow z\rangle + \beta|\Psi_{\rm part}(-g)\rangle|\downarrow z\rangle. \tag{90}$$

As the number of particle increases, $K \to \infty$, the macroscopic states $|\Psi_{\rm part}(+g)\rangle$ and $|\Psi_{\rm part}(-g)\rangle$ become orthogonal for any finite shift g,

$$\langle \Psi_{\rm part}(+g)|\Psi_{\rm part}(-g)\rangle = \left[\int \psi^*(x-g)\psi(x+g){\rm d}x\right]^K \overset{K\to\infty}{\longrightarrow} 0, \tag{91}$$

since the modulus of integral in the rhs is less than unity, for any choice of $\psi(x)$, and not just for the Gaussian one made in equation (89). Then the mixed state of the spin is given by equation (88), and according to the basic rule of quantum mechanics, for anyone dealing *only* with the spin in the future, will receive it pointing up with a probability $|\alpha|^2$, or pointing down, with a probability $|\beta|^2$. This is also true for the macroscopic pointer, which the Observer will receive in one of the two orthogonal states $|\Psi_{\rm part}(\pm g)\rangle$, with the same probabilities. The only thing that matters to the Observer, involved *only* with the pointer, is the state of the particles, and not how this state was created. The same statistical ensemble could be provided by an Alice, who flips a skewed coin, and depending on how it comes up, sends to Observer the pointer in a state $|\Psi_{\rm part}(+g)\rangle$, or $|\Psi_{\rm part}(+g)\rangle$, with the same probabilities $|\alpha|^2$ and $|\beta|^2$. In each case, he/she/it will see the pointer's COM, displaced to the left, or to the right, as discussed in section 7.

We note here a similarity with the case of consecutive measurements of the spin's direction, made by a set of inaccurate microscopic pointer's, which fire one after another [23]. Eventually, the spin ends up driven into one of the two possible states, and the information about which one is encoded not into position of an individual pointer, but into a collective variable, similar to the position of the COM. Thus, the standard quantum approach remains consistent for as long as the Observer has direct access only to macroscopic pointers in coordinate space.

We can add another Observer, also 'looking' at the same macroscopic pointer, by simply replacing in equation (83) $G(f - \sum_n x_n)$ with a product $G(f - \sum_n x_n)G(f' - \sum_n x_n)$. After tracing out the spin's and the particles's variables, we obtain a joint probability distribution, $\rho(f_{\rm COM}, f'_{\rm COM})$, representing a binary choice: both observers see the macroscopic pointer shifted by g either to the right, or to the left. (Note that the form of equation (83) prevents possible disagreements, provided $G(f-g)$ and $G(f+g)$ do not overlap). The state of the macroscopic pointer remains practically unchanged, so that other Observers can confirm the results of the first two, if they wish. With this we achieve the aim (D) of the Introduction.

10. Macroscopic pointer in a 'grotesque' state

Quantum mechanics allows macroscopic superpositions although in practice creating such superpositions may be difficult. Below we question what Observer would see when looking at a 'grotesque' [24] state, where all K particles, which make up the pointer, are prepared in one of the two Gaussian states, $\psi(x \pm d)$, well separated by a distance $2d$. The overall state is, therefore, similar to (83), except for the absence of the spin states, since this time no spin is involved

$$\left\langle f|\Phi_1\right\rangle = \int \mathrm{d}\underline{x}\; G(f-\sum_k x_k)(\alpha|\Psi_{\rm part}(\underline{x},+d)\rangle + \beta|\Psi_{\rm part}(\underline{x},-d)\rangle). \tag{92}$$

As before, the Observer detects the position of the pointer's COM. Noting that $\langle\Psi_{\rm part}(\underline{x}\pm d)|\Psi_{\rm part}(\underline{x}'\mp d)\rangle \simeq \delta(\underline{x}-\underline{x}')\prod_{k=1}^{K}|\psi(x_k\mp d)|^2$ and $\langle\Psi_{\rm part}(\underline{x},-d)|\Psi_{\rm part}(\underline{x}',+d)\rangle \approx 0$, and tracing out the pointer's variables, for the Observer's density matrix we obtain

$$\hat{R}_{\rm obs}(f,f') = \int \mathrm{d}\underline{x}\; G\left(f'-\sum_k x_k\right)G\left(f-\sum_k x_k\right)\times\left(|\alpha|^2\prod_{k=1}^{K}|\psi(x_k-d)|^2 + |\beta|^2\prod_{k=1}^{K}|\psi(x_k+d)|^2\right). \tag{93}$$

Choosing, as in section 10

$$\sigma_x/\sqrt{K} \ll \Delta f_{\rm COM} = \Delta f/K \ll d, \tag{94}$$

and sending $K\to\infty$, in the COM variables we find

$$\hat{R}_{\rm obs}(f_{\rm COM},f'_{\rm COM}) \sim |\alpha|^2\sqrt{\delta(f_{\rm COM}-d)\delta(f'_{\rm COM}-d)} + |\beta|^2\sqrt{\delta(f_{\rm COM}+d)\delta(f'_{\rm COM}+d)}, \tag{95}$$

where $\sqrt{\delta(x)} \equiv \lim_{a\to 0}(\pi a^2)^{-1/4}\exp(-x^2/2a^2)$. Thus, the Observer's density matrix is nearly diagonal, and the two possible measured values are $\pm d$. We must, therefore, conclude that presented with a pointer in a macroscopic superposition, the Observer will find it either at d, or $-d$, with the probabilities given by the absolute squares of the amplitudes α and β in equation (92).

11. Classical limit of the measurement theory

Our discussion would be incomplete without mentioning the purely classical measurement, where a classical pointer is employed to determine a projection of a large classical angular momentum onto a given direction $\vec{n}$. Thus, we consider a system of K spins-1/2 in the state (16), a pointer consisting of N particles, all in a Gaussian state $|\Psi_0^{\rm point}\rangle = \prod_{n=1}^{N}|\psi_n\rangle$

$$\psi(x_n) = (2\pi\sigma_x^2)^{-1/4}\exp[-x_n^2/4\sigma_x^2], \tag{96}$$

and an Observer, whose internal resolution is Δf, $G(f) = (2\pi\Delta f^2)^{-1/4}\exp[-f^2/4\Delta f^2]$. With the help of equations (31)–(35), for $K\gg 1$, we find the pure state of the spins+pointer+observer to be

$$|\Phi_1\rangle = (2\pi K\sin^2\theta)^{-1/4}\int \mathrm{d}f\int \mathrm{d}\underline{x}\sum_{j=0}^{K} G\left(f-\sum_{i=1}^{N}x_n\right) \times\prod_{n=1}^{N}\psi(x_n-A_j^{\rm tot})\exp\left[-\frac{(A_j^{\rm tot}-K\cos\theta)^2}{4K\sin^2\theta}\right]|f\rangle\,|x_n\rangle\,|j\rangle, \tag{97}$$

where $A_j^{\rm tot} = 2j-K$. Now there are three Gaussian widths, one associated with the localisation of the wave function of the K spins, $2\sqrt{K}\sin\theta$, another σ_x, describing the states of the particles which form the pointer, and $\Delta f = N\Delta f_{\rm COM}$, which describes the Observer's own resolution. Our goal is to choose them in such a way that the Observer gets an accurate result without disturbing neither the pointer, nor the spins. Performing the Gaussian integrals, for the distribution of the Observer's readings we obtain

$$\rho(f_{\rm COM}) = (2\pi\delta f_{\rm COM}^2)^{-1/2}\exp\left[-\frac{(f_{\rm COM}-K\cos\theta)^2}{2\delta f_{\rm COM}^2}\right], \tag{98}$$

where

$$\delta f_{\rm COM}^2 = \Delta f_{\rm COM}^2 + \sigma_x^2/N + K\sin^2\theta, \tag{99}$$

and for the density matrix of the spin system we find

$$\langle j'|\hat{R}_{\rm spins}|\hat{j}\rangle \equiv \langle\Phi_1|j'\rangle\langle j|\Phi_1\rangle = \langle\Psi_0|j'\rangle\langle j|\Psi_0\rangle \exp\left[-\frac{N(j-j')^2}{2\sigma_x^2}\right]. \tag{100}$$

After interacting with the spins, the mixed state of the particles, forming the classical pointer, consists of the initial states $|\Psi_0^{\rm point}\rangle$ shifted by $A_j^{\rm tot} = 2j - K, j = 0,1,..,K$, each shift occurring with the probability $\sim\exp\left[-(A_j^{\rm tot} - K\cos\theta)^2/2K\sin^2\theta\right]$ (see section 6). Thus, with $\sigma_x \gg \sqrt{2K}\sin\theta$ we can consider the set of particles to be shifted by $K\cos\theta$ as a whole. The next step, observation of the cloud by an Observer, whose intrinsic resolution is Δf, was discussed in section 7. In particular, if $\Delta f_{\rm COM} \gg \sigma_x/\sqrt{N}$, the Observer will register the COM of the particles at $f = K\cos\theta$, and leave the state of the classical pointer almost unchanged. We note also that in equation (102) the difference $(j - j')^2$ must be of order of $K\sin^2\theta/2$, and with $\sigma_x^2 \gg NK\sin^2\theta/2$, the initial pure state of the spin system will also remain largely unchanged.

In summary, with $K, N \gg 1$, we find ourselves in the classical regime if

$$\sqrt{K} \ll \sigma_x/\sqrt{N} \ll \Delta f_{\rm COM} \ll K, \tag{101}$$

which can be satisfied, for instance, for $\sigma_x \sim K^{5/6}, N \sim K^{1/6}$, and $\Delta f_{\rm COM} \sim K^{5/6}$. Now we have classicality as defined in the Introduction. A classical pointer couples to a large classical angular momentum (spin) in such a manner that the Observer obtains the classical value of its projection with a negligible error. A different Observer can then measure the spin's projection on a different direction, and obtain the corresponding classical value, also with a negligible error. Finally, the second Observer can 'look' at the first Observer's pointer, and verify the result obtained by his predecessor. All the mentioned values 'exist' in the sense that they can be obtained any number of times without altering the states of either the spin or the pointer.

We conclude this section by writing down the classical equations of motion for the case where a classical pointer monitors a system, composed of a large number of particles. By the CLT (see section 5), to a good accuracy, we can ascribe to each of the two its position as well as its momentum, say, (p, x) and (X, P), respectively. The measured operator $\hat{A}$ is replaced by a classical dynamical variable, $A(x, p)$, and the coupling with the pointer takes a form $g(t)PA(x, p)$, where $g(t)$ is some switching function. Now the Hamilton's equations read

$$\begin{aligned}
\partial_t x &= \partial_p H_0(x, p) + g(t)P\partial_p A(x, p),\\
\partial_t p &= -\partial_x H_0(x, p) - g(t)P\partial_x A(x, p),\\
\partial_t X &= g(t)A(x, p),\\
\partial_t P &= 0,
\end{aligned} \tag{102}$$

where $H_0(x, p)$ is the system's own energy in a state, characterised by x and p. The pointer's momentum is conserved, $P = $ const, and its displacement after a time T is

$$X(T) - X(0) = \int_0^T g(t)A(x(t, P), p(t, P))\mathrm{d}t, \tag{103}$$

which for a special choice $g(t) = 1/T = $ const coincides with a time average of the A. Note, however, that the pointer does not perturb the observed system if, and only if, the pointer's momentum is zero, $P = 0$. Quantally, an accurate determination of the initial position $X(0)$, leads to a large spread in the pointer's momenta around $P = 0$, which inevitably disturb the observed system. In the classical limit, it is possible to keep the last terms in the first and second of equations (102) small, compared to the large $H_0(x, p)$, and non-invasive monitoring of a classical system is, in principle, possible.

12. From classical back to quantum

Finally, we briefly return to the Observer, able to monitor macroscopic objects, by following their COM, to an accuracy Δf, as discussed in section 5. How can he/she/it learn that the classical description is, in fact, an approximation? There are at least two possible ways. One consists in studying small individual objects, such as a single spin-1/2. As discussed in the previous section, it is possible to prepare $K \gg 1$ spins all polarised along the z-axis, then obtain, practically with certainty, a zero value for the total spin's projection on the x-axis, and finally obtain the original value K, in another measurement along the z-axis. For a single spin, this is, of course, no longer true, as was discussed in section 10. Aligning it with the z-axis, one then obtains the value of $1/2$ or $-1/2$ along the x-axis, and has only a 50% chance to recover the original polarisation if a second measurement along the z-direction is made. Thus, the classical picture of an angular momentum having well defined components along all directions breaks down, and the Observer becomes aware of the different laws governing the behaviour of microscopic objects.

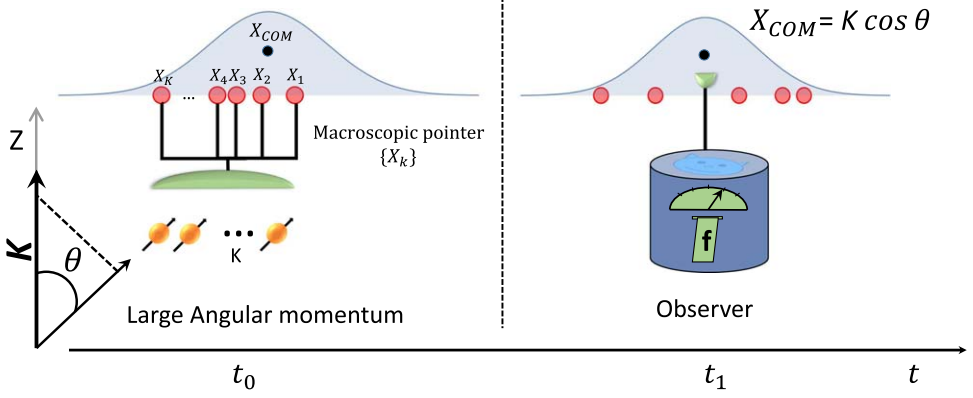

Figure 8. A macroscopic pointer consisting of $N \gg 1$ free particles in the same state and coupled to a large angular momentum $K \gg 1$, accurately measures the momentum's projection on the z-axis at t_1, without damaging its state. Later, at time t_2, an Observer determines the position of the pointer's centre of mass, also causing only a negligible damage to the state of the pointer. The measurement, if repeated, will yield the same result, which can be verified by other Observers.

Another possibility is taking a closer look at what so far had the appearance of a single macroscopic object. The Observer, whose ability to observe we have already reduced to that of a pointer monitoring the COM of a cloud of $K \gg 1$ quantum particles to an accuracy $\Delta f_{\rm COM}$, finds the COM at, say, the origin, $X_{\rm COM} = 0$. According to equation (65), for $\Delta f_{\rm COM} \gg \sigma_x/\sqrt{K}$ the error involved depends only on the Observer's own resolution. Furthermore, if the measurement is repeated after a time t, to an accuracy $\Delta f'$, such that (see equation (72))

$$\Delta f'^2_{\rm COM} \gg K^{-1}\left(2\sigma_x^2 + \frac{t^2}{2m^2\sigma_x^2}\right) + K^{-2}\frac{t^2}{2m^2\Delta f^2_{\rm COM}} \tag{104}$$

it will find the COM at the location predicted by classical mechanics, within the error bounds again determined by the Observer's resolution, $p_0 t/m - \Delta f'_{\rm COM} \lesssim X_{\rm COM} \lesssim p_0 t/m + \Delta f'_{\rm COM}$. See figure 8.

However, if the accuracy of the first measurement is significantly improved, $\Delta f_{\rm COM} \to 0$, so that now

$$\Delta f'^2_{\rm COM} \ll K^{-2}\frac{t^2}{2m^2\Delta f^2_{\rm COM}} \equiv \delta X^2_{\rm COM}, \tag{105}$$

the second measurement will find the COM of the cloud within a much broader range

$$p_0 t/m - \delta X_{\rm COM} \lesssim X_{\rm COM} \lesssim p_0 t/m + \delta X_{\rm COM}, \tag{106}$$

with the original cloud of particles dispersed by the interaction with the first pointer. This, in turn, will prompt the Observer to recognise that what he/she/it is dealing with is not an indivisible point-size object, obeying classical laws of motion, but a conglomerate of essentially quantum particles.

13. Conclusions and discussion

In summary, we have followed [9, 10] in asking whether it is possible, in principle, to obtain a classical picture by observing, not too scrupulously, macroscopic objects, composed of a large number of constituent parts, which individually obey known quantum laws. In particular, we studied a scheme in which a rudimentary Observer has a direct access to the position of a macroscopic pointer, which, in turn may be coupled to elementary quantum systems, such as a qubit, or to other classical-like systems, like large collections of polarised spins. We found it plausible that such an observer would be able to have an optimal resolution, which would allow him/her obtain a classical result to a high relative accuracy, while dealing with 'large' macroscopic systems, whose number of constituent parts, K, is of order of the Avogadro constant. Within these resolution limits, large conglomerates of quantum particles would be perceived as point-sized objects, moving along trajectories prescribed by the laws of classical mechanics. Such an object would appear to possess a well defined momentum, which can be measured to a good accuracy, without affecting the object's position. In a similar way, a large angular momentum would also be seen as classical quantity, all of whose components could be determined simultaneously, and with a small relative error. Should the measurements of this type be the only possible way to access physical reality, this reality would have an essentially classical aspect, with the underlying quantum nature revealed only through more accurate measurements, or through studying individual particles, or spins.

Such classical-like description is made possible by the tendency of the wave function, describing a macroscopic object, to become concentrated in a relatively small region of the spectrum of an operator representing a particular collective variable. This property is repeated in different representations, and for various non-commuting operators. Consequently, it is possible to devise a quantum measurement [9], which would not destroy the coherence between the essential components of a macroscopic quantum state, and yet yield only an error small compared to the large overall value of the measured quantity. With the state perturbed only slightly, each new measurement is no longer affected by its predecessors, and the measured values, associated with the peaks of the wave function, acquire certain objectivity, reminiscent of Einstein's 'elements of reality' [22].

Needless to say, our model is vastly oversimplified. Macroscopic object are not, in general, composed of non-interaction particles in Gaussian states. Spins do not usually all point in the same direction. A human eye observing the needle of a metre cannot be expected to perform a von Neumann measurement of the needle's COM. To estimate the COM's position an Observer would need to know also the number of particles, which make up the pointer. However, it retains some of the features, associated with everyday experiences. An Observer cannot directly 'see' a spin, or an angular momentum, yet is able to perceive large macroscopic objects in the coordinate space, clearly preferred by the human eye. A macroscopic pointer is perceived as a single object, rather than a point in a K-dimensional configuration space, as was first proposed in section 3. Our analysis requires no special role for human conscience [25] beyond what has been just said. Neither have we attempted to resolve the vexed problem of the 'wave function collapse' (see, for example, [26]), as we never tried to follow the fate of the parts of the quantum state, which correspond to the outcomes, not materialised in the course of the experiment. Rather we note that the Observer is involved only with the 'pointer' part of the spin+pointer composite. The state of the pointer in the entangled state, formed after the interaction with the spin, is a mixed one, which, already contains the probabilities. According to standard quantum mechanics, these probabilities are passed on to the Observer, who has no difficulty in seeing the COM of the pointer moved either to the left, or to the right, in each run of the experiment. In this imperfect way, we are able to place the divide between quantum and classical behaviour at the stage where the measured value becomes encoded in the displacement of a macroscopic object with classical properties. It remains to be seen whether the model, despite of its numerous shortcomings, captures the essential features of the quantum-to-classical behaviour. Yet, as was claimed in [9], it offers an alternative way towards its understanding.

We conclude with an attempt at a more general excuse for leaving outside any mention of Observer's conscience. In our discussion, an Observer has access to only a fraction of the complex world, through his/her/its ability to perceive certain 'physical phenomena'. The 'physical laws', established by the Observer, are, therefore, but connections and relations between Observer's accessible experiences. As such, they may not amount to an exhaustive objective picture of the world, but are destined to reflect particular properties and limitations of the Observer itself. Such is, for example, the rule (see section 7) that the COM of a large cluster of free particles always moves with a constant velocity. The very notion of COM is particular to the manner in which the Observer perceives this macroscopic object. The 'unseen' part of the far more complex world will remain outside the remit of Observer's physics, unless or until a better experiment provides for a new experience. It is possible then that new physics, based on this experience (e.g. quantum theory), will still be incomplete, yet leave room for further improvement. It may also be possible that Observer's own abilities to perceive will have been exhausted, without providing a complete understanding of physical reality. In this latter case, a theory is likely to descend, like it seems to happen with quantum mechanics, to the level where 'no one has found any machinery behind the law' [27]. An Observer, aware of own limitations, will almost certainly consider him/her/itself not a suitable object of a study, as has often been claimed in modern literature (see, for example, [28]). With the Observer excluded from the list of the studied physical phenomena, quantum mechanics acquires internal consistency. Outcomes of the experiments an Observer may perform on the parts of the outside world occur with certain probabilities, or frequencies. Quantum theory provides the recipes for calculating those probabilities for all eventualities.

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DS, SB and DA wrote the paper, and reviewed it.

Competing interests

The authors declare no competing interests.

Appendix A

Consider a quantum system with a continuum spectrum, and its three possible bases $\{|\mu\rangle\}$, $\{|\nu\rangle\}$ and $\{|\gamma\rangle\}$ such that

$$\langle z|z'\rangle = \delta(z-z'), \int \mathrm{d}z|z\rangle\langle z| = \hat{I}, \text{ with } z \in \{\mu, \nu, \gamma\}. \tag{107}$$

For a composite consisting of K such systems we can construct three complete product bases

$$|\underline{z}\rangle \equiv \prod_{i=1}^{K} |z_i\rangle, \ \langle \underline{z}|\underline{z}'\rangle = \prod_i \delta(z_i - z_i') \equiv \delta(\underline{z}-\underline{z}'). \tag{108}$$

We will also need two operators, with eigenstates $|\underline{\mu}\rangle$ and $|\underline{\nu}\rangle$,

$$\hat{A} = \int \mathrm{d}\underline{\mu}|\underline{\mu}\rangle A(\underline{\mu})\langle\underline{\mu}|, \ \hat{B} = \int \mathrm{d}\underline{\nu}|\underline{\nu}\rangle B(\underline{\nu})\langle\underline{\nu}|, \tag{109}$$

where $\int \mathrm{d}\underline{z} = \int \mathrm{d}z_1 \ldots \mathrm{d}z_k$, and $A(\underline{\mu}) \equiv A(\mu_1,\ldots,\mu_K)$, etc. We will be interested in the case when $\hat{A}$ and $\hat{B}$ are sums of variables describing individual subsystems, i.e.

$$A(\underline{\mu}) = \sum_{i=1}^{K} A_i(\mu_i), \ B(\underline{\nu}) = \sum_{i=1}^{K} B_i(\nu_i). \tag{110}$$

Now we can define the probability amplitude for the composite to start in some state $|\Psi_0\rangle$, 'pass through' a state $|\underline{\mu}\rangle$ at $t = 0$, evolve until $t > 0$, pass through $|\underline{\nu}\rangle$, and end up in $|\underline{\gamma}\rangle$,

$$\mathcal{A}(\underline{\gamma} \leftarrow \underline{\nu} \leftarrow \underline{\mu} \leftarrow \Psi_0) = \langle\underline{\gamma}|\underline{\nu}\rangle\langle\underline{\nu}|\hat{U}(t)|\underline{\mu}\rangle\langle\underline{\mu}|\Psi_0\rangle, \tag{111}$$

where $\hat{U}(t) = \exp(-\mathrm{i}\hat{H}t)$ is the composite's evolution operator. With this we can define the amplitude for having $\hat{A} = a$, and then $\hat{B} = b$, before arriving in $|\underline{\gamma}\rangle$,

$$\mathcal{A}(\underline{\gamma} \leftarrow b \leftarrow a \leftarrow \Psi_0) = \int \mathrm{d}\underline{\mu}\ \mathrm{d}\underline{\nu}\ \langle\underline{\gamma}|\underline{\nu}\rangle\ \delta(B(\underline{\nu}) - b)\langle\underline{\nu}|\hat{U}(t)|\underline{\mu}\rangle\,\delta(A(\underline{\mu}) - a)\langle\underline{\mu}|\Psi_0\rangle. \tag{112}$$

If two von Neumann pointers, prepared in Gaussian states, $G(f)$ and $G'(f')$, are employed to measure $\hat{A}$ and $\hat{B}$, the absolute square of the coarse grained amplitude

$$\mathcal{A}(\underline{\gamma} \leftarrow f' \leftarrow f \leftarrow \Psi_0) = \int \mathrm{d}a\ \mathrm{d}b\ G(f'-b)G(f-a)\mathcal{A}(\underline{\gamma} \leftarrow b \leftarrow a \leftarrow \Psi_0). \tag{113}$$

does yield the joint probability to have pointer readings f and f', and find the system in the final state $|\underline{\gamma}\rangle$, $\rho(f, f'|\underline{\gamma}) = |\mathcal{A}(\underline{\gamma} \leftarrow f' \leftarrow f \leftarrow \Psi_0)|^2$ [19]. Finally, summing over all final states gives the probability to find the pointer readings f and f', $\rho(f, f') = \int \mathrm{d}\underline{\gamma}\ \rho(f, f'|\underline{\gamma})$. It can also be written as the square of the norm of a state,

$$\rho(f, f') = \langle\Phi(f, f')|\Phi(f, f')\rangle, \tag{114}$$

where

$$|\Phi(f, f')\rangle = \hat{G}_B'(f')\hat{U}(t)\hat{G}_A(f)|\Psi_0\rangle, \tag{115}$$

and we have introduced operators

$$\begin{aligned} \hat{G}_A(f) &\equiv \int \mathrm{d}\underline{\mu}|\underline{\mu}\rangle G(f - A(\underline{\mu}))\langle\underline{\mu}|, \\ \hat{G}_B'(f') &\equiv \int \mathrm{d}\underline{\nu}|\underline{\nu}\rangle G'(f' - B(\underline{\nu}))\langle\underline{\nu}|, \end{aligned} \tag{116}$$

or, explicitly,

$$\rho(f, f') = \int \mathrm{d}\underline{\nu}\ G'^2(f' - B(\underline{\nu}))|\langle\Psi_0|\hat{G}_A(f)\hat{U}(t)|\underline{\nu}\rangle|^2. \tag{117}$$

Appendix B

Bearing in mind that $\int_{-\infty}^{\infty}\exp\left[-x^2/a^2+bx\right]\mathrm{d}x=\sqrt{\pi a^2}\exp[]a^2b^2/4]$, it is easy to evaluate the following integrals ($\mathrm{d}\underline{x}\equiv \mathrm{d}x_1 ..., \mathrm{d}x_N$).

$$\begin{aligned}I(f,A,B,a,b,c,N)&\equiv B\int \mathrm{d}\underline{x}\exp\left[-\left(f-\sum_{\mathrm{i}=1}^{N}x_{\mathrm{i}}/c\right)^2/b^2\right]\prod_{j=1}^{N}A\exp[-x_j^2/a^2]\\&=\sqrt{b^2/4\pi}A^NB\int \mathrm{d}\lambda\exp[-b^2\lambda^2/4+\mathrm{i}\lambda f]\prod_{j=1}^{N}\int \mathrm{d}x_j\exp[-x_j^2/a^2-\mathrm{i}\lambda x_j/c]\\&=\sqrt{b^2/4\pi}A^NB(\pi a^2)^{N/2}\int \mathrm{d}\lambda\exp\left[-\lambda^2(b^2+a^2N/c^2)/4+\mathrm{i}\lambda f\right]\\&=A^NB\sqrt{\frac{\pi^Na^{2N}b^2}{b^2+Na^2/c^2}}\exp\left[-f^2/(b^2+a^2N/c^2)\right].\end{aligned}\tag{118}$$

We will also require a ratio

$$\begin{aligned}R(f)\equiv\frac{I(f,A,1,a,b,N,N)}{I^{1/2}(f,A,1,a,b/\sqrt{2},N,N)}&=\frac{(A^{2N}\pi^Na^{2N}b^2(b^2+2a^2/N)^{1/4}}{(b^2+a^2/N)^{1/2}}\\&\times\exp\left[-\frac{a^2f^2}{N(b^2+a^2/N)(b^2+2a^2/N)}\right]\end{aligned}\tag{119}$$

We note that for $A=(\pi a^2)^{-1/2}$, $B=(\pi b^2)^{-1/2}$, and $N\to\infty$, $I(f,A,B,a,b,N,N)$ tends to a Gaussian distribution of a width b

$$I(f,A,B,a,b,N,N)\to(\pi b^2)^{-1/2}\exp[-f^2/b^2],\tag{120}$$

and

$$R(f)\to\exp[-a^2f^2/Nb^4].\tag{121}$$

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