Quantum decoherence of an anharmonic oscillator monitored by a Bose-Einstein condensate

D. Alonso* and S. Brouard†

Instituto Universitario de Estudios Avanzados (IUdEA) and Departamento de Física, Universidad de La Laguna, La Laguna E38204, Tenerife, Spain

D. Sokolovski‡

Department of Physics Chemistry, University of the Basque Country, Leioa, Spain and IKERBASQUE, Basque Foundation for Science, 48011, Bilbao, Spain

(Received 25 March 2014; published 11 September 2014)

The dynamics of a quantum anharmonic oscillator whose position is monitored by a Bose-Einstein condensate trapped in a symmetric double well potential is studied. The (nonexponential) decoherence induced on the oscillator by the measuring device is analyzed. A detailed quasiclassical and quantum analysis is presented. In the first case, for an arbitrary initial coherent state, two different decoherence regimes are observed: an initial Gaussian decay followed by a power law decay for longer times. The characteristic time scales of both regimes are reported. Analytical approximated expressions are obtained in the full quantum case where algebraic time decay of decoherence is observed.

DOI: 10.1103/PhysRevA.90.032106 PACS number(s): 03.65.Yz, 03.65.Ta, 03.75.Gg

I. INTRODUCTION

Recent experiments [1,2] show that a great degree of coherent control is possible between micromechanical oscillators and a Bose-Einstein condensate (BEC) of magnetically trapped rubidium-87 atoms [3]. Chip-based magneto traps offer a high degree of control when the BEC and a micromechanical cantilever are brought close to distances of the order of the micrometer. At that level of proximity between a cantilever and a BEC, the surface forces start to play a role. Such forces allow one to couple coherently the collective dynamics of a condensate and a mechanical oscillator. Accordingly, it is possible to study the interaction between trapped atoms and on-chip-solid-state systems such as nano- and micromechanical oscillators [4–6].

One of the major experimental goals is to use neutral atoms to coherently manipulate the state of the oscillator. There are several proposals aimed at achieving this by employing atoms in a cavity with a moving mirror [7–9], or by coupling atoms by means of a reflective membrane, where the lattice trapping the atoms is built by reflecting a laser beam off the membrane [10,11].

Such optomechanical systems, composed by nanomechanical oscillators and atoms interfaced via optical quantum buses, have been recently discussed in the context of quantum nondemolition Bell measurements and the ability to prepare Einstein-Podolsky-Rosen entangled states [12]. Also ions are proposed as transducers for electromechanical oscillators [13] while other proposals involve the coupling between oscillators and dipolar molecules [14]. A growing interest, both theoretical and experimental, is therefore apparent in the study of nanomechanical oscillators and their interaction with other quantum systems [4,15]. In these systems it is possible to achieve different levels of coherent control by incorporating them into combined (hybrid) devices, involving single electron transistors [16,17] and point contacts (PCs) [18], microwave cavities in superconducting regime [19], or superconducting qubits [20,21]. These numerous experiments and theoretical proposals indicate the feasibility of studying quantum correlations, quantum control of mechanical force sensors, and decoherence in the regime where strong coherent coupling is achieved.

In particular, the experimental advances mentioned above will enhance our ability to test fundamental quantum properties, such as decoherence in a well controlled setting [1]. The determination of decoherence rates to a high accuracy, and their comparison to theoretical predictions will be possible in the near future. One particularly interesting goal would be to explore the quantum-to-classical transition [22] in the dynamics of the mechanical oscillator, and the possibly anomalous decoherence that the oscillator may exhibit when in contact with a BEC.

In [23] the dynamics of an oscillator coupled to a BEC trapped in a symmetric double-well potential, with the atomic current dependent on the oscillator coordinate was analyzed. The fact that the bosons tunnel into a single state, rather than into a broad energy zone, as in the case of a point contact, gives the decoherence process unusual properties. Thus, a qubit monitored by a BEC undergoes an anomalously slow state-dependent decoherence [24], while its decoherence in the presence of a PC is exponential in time. Similarly, a harmonic quantum oscillator whose position is being monitored is capable of retaining some, or even all, of its coherence [23]. One of the reasons for such behavior lies in the fact that a displaced harmonic oscillator maintains its equidistant level structure, and its motion remains periodic even when coupled to a BEC via its position. This may not be true if there is even a small degree of anharmonicity in the oscillator's motion. The effect of anharmonicity on the decoherence rate of an oscillator coupled to a BEC trapped in a double-well structure is the subject of this work.

The rest of the paper is organized as follows. A brief description of the model is presented in Sec. II. A quasiclassical

*dalonso@ull.es
†sbrouard@ull.es
‡dgsokol15@gmail.com

analysis of the system dynamics and its implications in the appearance of decoherence is presented in Sec. III A. Section III B contains the results of a full quantum analysis. Our conclusions are presented in Sec. IV.

II. THE "GATEKEEPER" MODEL

We are interested in an anharmonic nanomechanical oscillator coupled to a BEC in such a way that its position influences the atomic current flowing between the wells of a double-well potential in which the BEC's atoms are trapped. Thus we consider, in one dimension, a quartic anharmonic oscillator of mass m and frequency ω_0, described by the Hamiltonian

$$\hat{H}_{\rm osc} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{X}^2 + \frac{\beta}{4}\hat{X}^4, \tag{1}$$

where β controls anharmonicity of the potential. The oscillator is coupled to a BEC composed by noninteracting bosonic atoms trapped in a symmetric double-well potential. Without tunneling the atoms may occupy the left or right well

$$\hat{H}_{\rm con} = E_0\,(\hat{c}_L^\dagger\hat{c}_L + \hat{c}_R^\dagger\hat{c}_R), \tag{2}$$

where the operator $\hat{c}_L^\dagger$ ($\hat{c}_R^\dagger$) creates a boson in the ground state of the left (right) well. We choose the coupling to be linear in the BEC's tunneling operator $\hat{T} = \hat{c}_L^\dagger\hat{c}_R + \hat{c}_R^\dagger\hat{c}_L$, and assuming oscillation to be small, linearize it also in the oscillator's position X,

$$\hat{H}_{\rm int} = \hbar\Omega(\hat{X})\otimes\hat{T} \approx \hbar[\Omega_0 + \Omega_1\hat{X}]\,(\hat{c}_L^\dagger\hat{c}_R + \hat{c}_R^\dagger\hat{c}_L). \tag{3}$$

Now with tunneling switched on, the full Hamiltonian is given by [24]

$$\hat{H} = \hat{H}_{\rm osc} + \hat{H}_{\rm con} + \hat{H}_{\rm int}. \tag{4}$$

Initially the system is prepared in a product state, and its density matrix has the form

$$\hat{\rho}(0) = \hat{\rho}_{\rm osc}(0)\otimes\hat{\rho}_{\rm con}(0), \tag{5}$$

where $\hat{\rho}_{\rm con}(0)$ corresponds to some nonequilibrium state of the BEC. The flow of bosons across the barrier dividing the wells depends on $\hat{X}$, and may be used to extract information about the oscillator's position. The back action provided by such a measurement on the observed oscillator is the subject of this paper. A sketch of a possible experimental setup is shown in Fig. 1.

Since the trap is symmetric, the tunneling operator commutes with the $\hat{H}_{\rm con}$, $[\hat{H}_{\rm con},\hat{T}] = 0$. Suppose the BEC is prepared in a stationary state $|\tilde{\phi}_n\rangle$,

$$\begin{aligned}|\tilde{\phi}_n\rangle &= [2^N(N-n)!n!]^{-1/2}(\hat{c}_L^\dagger + \hat{c}_R^\dagger)^{N-n}(\hat{c}_L^\dagger - \hat{c}_R^\dagger)^n|0\rangle_{\rm con}\\ &\quad (n = 0,1,\ldots,N),\\ \hat{c}_L|0\rangle_{\rm con} &= \hat{c}_R|0\rangle_{\rm con} = 0,\end{aligned} \tag{6}$$

for which we also have $\hat{H}_{\rm con}|\tilde{\phi}_n\rangle = NE_0$ and $\hat{T}|\tilde{\phi}_n\rangle = (N-2n)|\tilde{\phi}_n\rangle$. It is readily seen that the BEC will continue in $|\tilde{\phi}_n\rangle$, while the oscillator would experience a constant energy shift of $\hbar\Omega_0(N-2n)$ and an additional force $\varphi_n \equiv \hbar\Omega_1(N-2n)$. Moreover, the density matrix of the oscillator at a time t, $\hat{\rho}_{\rm osc}(t) = {\rm Tr}_{\rm con}\{\rho(t)\}$, is given by a weighted sum of density matrices $\hat{\rho}_{\rm osc}^{(n)}$ evolved from $\hat{\rho}_{\rm osc}(0)$ under different forces φ_n,

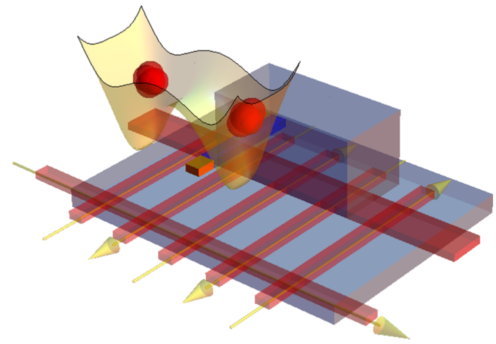

FIG. 1. (Color online) Graphical representation of an atom chip that can be a realization of the model studied in this work. An array of wires is used to create two dimple traps that contain atoms (red). The traps are separated by tunable barriers. The yellow arrows indicate the currents used to create the traps. On top of the wire configuration that supports the traps, the cantilever (blue) is mounted with a tip (orange) that interacts with the atoms. The figure is inspired by the work [3].

$$\hat{\rho}_{\rm osc}(t) = \sum_n P(n)\,\hat{\rho}_{\rm osc}^{(n)}(t), \tag{7}$$

where

$$\begin{aligned}\hat{\rho}_{\rm osc}^{(n)}(t) &\equiv e^{-i\hat{\mathcal{H}}_{\rm osc}(\varphi_n)t/\hbar}\hat{\rho}_{\rm osc}(0)e^{i\hat{\mathcal{H}}_{\rm osc}(\varphi_n)t/\hbar},\\ \hat{\mathcal{H}}_{\rm osc}(\varphi_n) &\equiv \hat{H}_{\rm osc} + \varphi_n\hat{X},\end{aligned} \tag{8}$$

and $P(n)$ is the probability to find the BEC in a state $|\tilde{\phi}_n\rangle$ at $t = 0$,

$$P(n) = {\rm Tr}_{\rm con}[|\tilde{\phi}_n\rangle\langle\tilde{\phi}_n|\hat{\rho}_{\rm con}(0)]. \tag{9}$$

(We note that the constant energy shifts cancel, and do not contribute to the oscillators evolution.)

Similarly, for the expectation value of an oscillator's observable $\hat{O}$, we have

$$\begin{aligned}\langle\hat{O}\rangle &= \sum_n P(n){\rm Tr}_{\rm osc}\big[\hat{\rho}_{\rm osc}^{(n)}(t)\hat{O}\big]\\ &= \sum_n P(n){\rm Tr}_{\rm osc}[\hat{\rho}_{\rm osc}(0)\hat{O}_n(t)],\end{aligned} \tag{10}$$

with $\hat{O}_n(t) = e^{i\hat{\mathcal{H}}_{\rm osc}(\varphi_n)t/\hbar}\hat{O}e^{-i\hat{\mathcal{H}}_{\rm osc}(\varphi_n)t/\hbar}$.

We can consider a limit in which the number of bosons in the BEC increases, while the individual tunneling probability is reduced, so that there is a finite atomic current between the two wells,

$$N \to \infty, \qquad \Omega_{0,1} \to 0, \qquad \Omega_{0,1}\sqrt{N} = \kappa_{0,1}. \tag{11}$$

Preparing all the atoms in the left well,

$$\hat{\rho}_{\rm con}(0) = |\psi_0\rangle\langle\psi_0|, \qquad |\psi_0\rangle \equiv (\hat{c}_L^\dagger)^N|0\rangle_{\rm con}/\sqrt{N!}, \tag{12}$$

we have a source of practically irreversible current, since the Rabi period after which the BEC returns to its initial state is now very large [24]. Measuring after a time t the number of atoms in the right well gives information about the oscillator's past [25]. Also the sums in Eqs. (7) and (10) can be replaced by integrals, $\sum_n P(n) \to \int d\varphi\, P(\varphi)$. Continuous distribution of forces corresponding to the initial state (12) is Gaussian,

$$P(\varphi) = \frac{e^{-\varphi^2/2\Delta_\varphi^2}}{\sqrt{2\pi\,\Delta_\varphi^2}}, \tag{13}$$

with $\Delta_\varphi^2 \equiv 2m\hbar\omega_0\kappa^2$ [24].

III. MONITORING POSITION OF A QUARTIC ANHARMONIC OSCILLATOR

The process of decoherence appears in general in the dynamics of averages of particular observables. Then, instead of analyzing the density matrix (7), it is convenient to consider the oscillator's mean position, thus choosing the operator $\hat{O}$ in Eq. (10) as the operator $\hat{X}$,

$$\langle \hat{X}(t)\rangle = \sum_n P(n)\langle \hat{X}_n(t)\rangle,$$

which reads, in the continuous limit,

$$\begin{aligned}\langle \hat{X}(t)\rangle &= \int_{-\infty}^{\infty} d\varphi\;\, P(\varphi)\langle \hat{X}_\varphi(t)\rangle \\ &= \int_{-\infty}^{\infty} d\varphi\;\, P(\varphi)\bigg\{\sum_i \langle\psi_i^\varphi|\hat\rho_{\rm osc}(0)|\psi_i^\varphi\rangle\langle\psi_i^\varphi|\hat{X}|\psi_i^\varphi\rangle \\ &\quad + \sum_{i\neq j} e^{-i(E_i^\varphi - E_j^\varphi)t/\hbar}\langle\psi_i^\varphi|\hat\rho_{\rm osc}(0)|\psi_j^\varphi\rangle\langle\psi_j^\varphi|\hat{X}|\psi_i^\varphi\rangle\bigg\},\end{aligned} \tag{14}$$

with $\hat{\mathcal{H}}_{\rm osc}(\varphi)|\psi_i^\varphi\rangle = E_i^\varphi|\psi_i^\varphi\rangle$. Indeed, it can be observed that in the case $E_i^\varphi - E_j^\varphi \neq \text{const}(\varphi)$, the exponentials in the second term of Eq. (14) are rapidly oscillating functions and consequently such term will vanish. Therefore $\hat\rho_{\rm osc}(t)$ [and with it the averages (10)] will tend to stationary values as $t\to\infty$. Without such a cancellation, the oscillator will not be able to reach a steady state no matter how long one waits.

Equation (14) is the starting point of the quantum calculations we shall present.

A. Time evolution in Wigner space

We shall first look at the time evolution of a state of the oscillator in Wigner representation when it is influenced by the condensate. In that representation, the state develops structure as time progresses. An initial state of the oscillator evolves according to a Schrödinger equation that takes into account the effect of the condensate. In one dimension the Wigner function associated to a quantum state $\hat\rho$ is defined as [26]

$$W(x,p,t) = \frac{1}{2\pi\hbar}\int dq\; e^{ipq/\hbar}\left\langle x-\frac{q}{2}\right|\hat\rho\left|x+\frac{q}{2}\right\rangle. \tag{15}$$

It is convenient to write a state $\hat\rho$ of the oscillator for a given time t in terms of the eigenstates $|\varphi_n\rangle$ of the harmonic oscillator as

$$\hat\rho(t) = \sum_{n,m} c_{nm}(t)|\varphi_n\rangle\langle\varphi_m|. \tag{16}$$

By inserting this expression in (15) it follows the Wigner function corresponding to the state of the oscillator as

$$W(x,p,t) = \sum_{n,m} c_{nm}(t) w_{nm}(x,p), \tag{17}$$

where

$$w_{nm}(x,p) = \frac{1}{2\pi\hbar}\int\; e^{ipq/\hbar}\left\langle x-\frac{q}{2}\right|\varphi_n\Big\rangle\Big\langle\varphi_m\left|x+\frac{q}{2}\right\rangle. \tag{18}$$

The integral in this expression has a representation in terms of generalized Laguerre polynomials L_n^a yielding

$$w_{nm}(x,p) = \begin{cases} \frac{(-1)^n e^{-|z|^2}}{\pi\hbar}\left(\frac{n!\,2^m}{m!\,2^n}\right)^{1/2}(z^*)^{m-n}L_n^{m-n}(2|z|^2) & \text{if } n\leqslant m \\ \frac{(-1)^m e^{-|z|^2}}{\pi\hbar}\left(\frac{m!\,2^n}{n!\,2^m}\right)^{1/2}(z)^{n-m}L_m^{n-m}(2|z|^2) & \text{if } n\geqslant m, \end{cases} \tag{19}$$

with $z = x\sqrt{m\omega_0/\hbar} + ip\sqrt{\hbar m\omega_0}$. When Eq. (19) is combined with Eq. (17) it follows an analytical expression for the Wigner function of the oscillator. In Fig. 2 the Wigner function of the oscillator at different times is plotted. It can be seen that as time proceeds an initial coherent state starts to spread over phase space developing a highly oscillatory structure, that at the end is responsible for the decay of the position expectation value. The study of such decay and of its details is precisely the subject of this work.

B. Quasiclassical approximation

We begin by evaluating the quasiclassical limit of (14) which is conveniently obtained by employing the Weyl-Wigner representation [26,27]. For the average of an arbitrary operator describing the oscillator, $\hat{O}_n(t)$, we write

$$\langle\hat{O}_n(t)\rangle = \int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dp\; W(x,p)\,O_n(x,p,t), \tag{20}$$

where $W(x,p)$ and $O_n(q,p,t)$ are the Weyl-Wigner transforms of the initial oscillator's state, and of the operator $\hat{O}_n(t)$ in its Heisenberg representation,

$$O_n(x,p,t) = \int dq\; e^{ipq/\hbar}\left\langle x-\frac{q}{2}\right|\hat{O}_n(t)\left|x+\frac{q}{2}\right\rangle. \tag{21}$$

The dynamical aspects of the formalism are contained in the equation of motion for $O_n(x,p,t)$,

$$\partial_t O_n(x,p,t) = \{\{\mathcal{H}(x,p;\varphi_n), O_n(x,p,t)\}\}, \tag{22}$$

where $\mathcal{H}(x,p;\varphi_n)$ is the Wigner representation of the quantum Hamiltonian operator $\hat{\mathcal{H}}_{\rm osc}$, and the Moyal (sine) bracket is

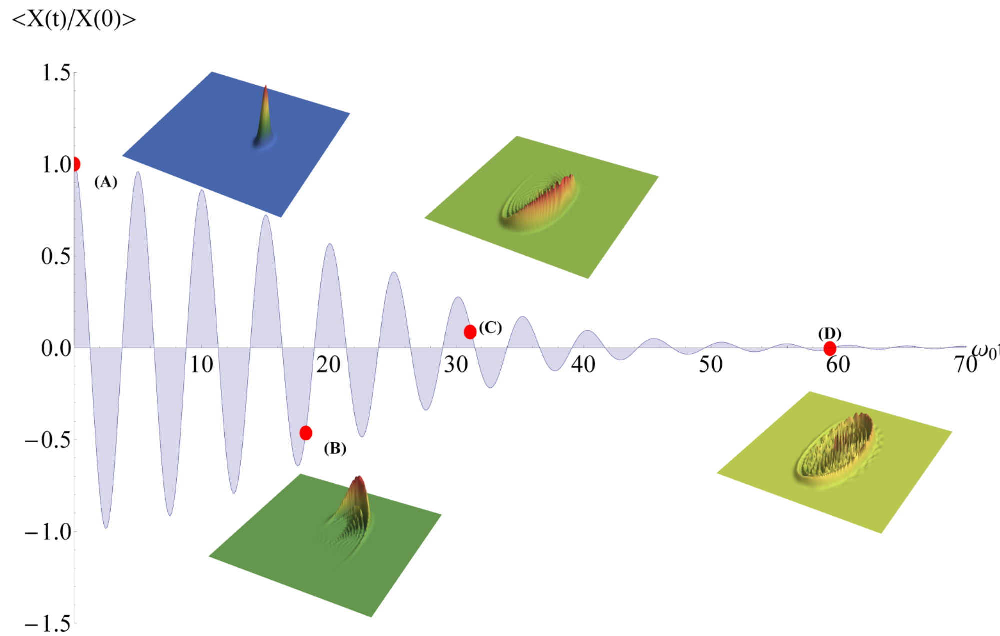

FIG. 2. (Color online) Decay of the expectation value of position and some Wigner functions of the oscillator at different times.

defined, as usual [28], by

$$\{\{f(x,p),g(x,p)\}\} \equiv \frac{2}{\hbar} f(x,p)\sin\left[\frac{\hbar}{2}(\overleftarrow{\partial}_p\overrightarrow{\partial}_x - \overleftarrow{\partial}_x\overrightarrow{\partial}_p)\right] g(x,p)$$

with, e.g., $f(x,p)\ \overleftarrow{\partial}_p\overrightarrow{\partial}_x\ g(x,p) \equiv \partial_p f(x,p)\partial_x g(x,p)$. Equations (20)–(22) are a convenient starting point for our quasiclassical analysis. Expanding $O_n(x,p,t) = X_n(x,p,t)$ in powers of $\hbar$, as

$$X_n(x,p,t) = X_n^{\rm cl}(x,p,t) + \hbar^2 X_n^{(2)}(x,p,t) + O(\hbar^4), \quad (23)$$

and recalling that as $\hbar \to 0$ the Moyal bracket reduces to the classical Poisson bracket $\{*,*\}$ [29], we obtain an approximate equation of motion for $X_n^{\rm cl}(x,p,t)$,

$$\partial_t X_n^{\rm cl}(x,p,t) = \{\mathcal{H}(x,p;\varphi_n), X_n^{\rm cl}(x,p,t)\}, \quad (24)$$

with the initial condition

$$X_n^{\rm cl}(x,p,t=0) = x, \quad (25)$$

which is obtained by evaluating the integral (21) for $t = 0$. A solution of the classical equation of motion (24) is any function $\mathcal{F}(x_t(x,p), p_t(x,p))$, provided x_t and p_t satisfy the Hamiltonian equations of motion (a dot denotes the time derivative)

$$\begin{aligned} \dot{p}_t &= -\partial_{x_t}\mathcal{H}(x_t,p_t;\varphi_n) = -m\omega_0^2 x_t - \beta x_t^3 - \varphi_n, \\ \dot{x}_t &= \partial_{p_t}\mathcal{H}(x_t,p_t;\varphi_n) = p_t/m, \end{aligned} \quad (26)$$

subject to $x_{t=0} = x$ and $p_{t=0} = p$. With the help of the initial condition (25) we identify $\mathcal{F}(x,p)$ with x, so that $X_n^{\rm cl}(x,p,t) = x_t(x,p)$. Thus, $X_n^{\rm cl}(x,p,t)$ is just the position, at a time t, of the oscillator whose initial position and momentum at $t = 0$ were x and p, respectively.

It is readily seen that in the quasiclassical limit, the task of calculating the mean oscillator's position at time t,

$$\langle \hat{X}(t)\rangle_{\rm qcl} = \sum_n P(n)\left[\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, W(x,p)\, X_n^{\rm cl}(x,p,t)\right], \quad (27)$$

reduces to choosing initial phase space distribution $W(x,p)$, which contains all quantum effects, and evaluating classical oscillator trajectories for different values of the induced force φ_n. In the limit of small anharmonicity ($\frac{\beta x_0^2}{\omega_0^2} \ll 1$) the system will be shown to behave as a set of harmonic oscillators, each one with a slightly shifted frequency, and the semiclassical approximation described by the previous equation is very accurate for the purpose of studying the decoherence effect induced by the coupling with the BEC system. To support the accuracy of the approximations made, comparisons with the results obtained by exact numerical integration of the Schrödinger equation are provided. The numerics give strong support to the semiclassical approximation in the limit we are considering.

C. A coherent initial state: Small anharmonicity

Next we specify our analysis to the case where the oscillator is prepared in a coherent state, whose Weyl-Wigner transform is given by

$$W(x,p) = \frac{1}{\pi\hbar} e^{-m\omega_0(x-x_0)^2/\hbar - (p-p_0)^2/m\hbar\omega_0}. \quad (28)$$

There are no analytical solutions for a classical anharmonic oscillator. However, the decoherence effects absent for a harmonic oscillator, appear already in the limit of small anharmonicity ($\frac{\beta x_0^2}{\omega_0^2} \ll 1$). In this case approximate oscillator trajectories can be obtained, e.g., by the method of strained coordinates (Lindsted-Poincaré method) for periodic solutions [30], which we will describe here briefly. We begin by considering a trajectory such that at some $t = t_0$ it passes through some x^0 with a zero momentum, $x_{t_0} = x^0$, $p_{t_0} = 0$. This can be represented by a sum of harmonic functions with phases and amplitudes modified at different orders in β. An approximate solution to the first order in β for the phase and to zero order for the amplitude is given by [30,31]

$$X_n^{\rm cl}(x^0,0,t) = -\frac{\varphi_n}{m\omega_0^2} + \left(\frac{\varphi_n}{m\omega_0^2} + x^0\right) \times \cos\{\omega_0[1+\beta\Delta(x^0,0;\varphi_n)](t-t_0)\}, \quad (29)$$

with

$$\Delta(x,p;\varphi_n) = \frac{3}{4\left(m\omega_0^2\right)^2}\,\mathcal{H}_0(x,p;\varphi_n) + \frac{15}{8m^3\omega_0^6}\,\varphi_n^2 + O(\beta), \quad (30)$$

and

$$\mathcal{H}_0(x,p;\varphi_n) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 + \varphi_n x. \quad (31)$$

The solution corresponding to an arbitrary choice of initial x and p is then obtained by choosing in Eq. (29) x^0 and t_0 in such a way that the trajectory specified by Eq. (29) would, at $t = 0$, pass through x with the desired momentum p. Explicitly, we have

$$X_n^{\rm cl}(x,p,t) = -\frac{\varphi_n}{m\omega_0^2} + \left(\frac{\varphi_n}{m\omega_0^2} + x\right)\cos(\omega_1 t) + \frac{p}{m\omega_0}\sin(\omega_1 t), \quad (32)$$

which describes a harmonic motion whose frequency is modified both by the anharmonicity of the oscillator potential and the presence of the BEC, and also depends on the initial position x and momentum p of the oscilllator,

$$\omega_1 = \omega_0[1+\beta\Delta(x,p;\varphi_n)].$$

Higher order corrections in β can be systematically obtained if necessary, although decoherence for small enough β ($\frac{\beta x_0^2}{\omega_0^2} \ll 1$) is accurately described at this level of approximation. In particular a third harmonic contribution, with an amplitude which is first order in β, is negligible as compared to the decoherence and dephasing effects we will show next to be produced by the combined effect of frequency shift in the first harmonic (which is also first order in β) and the interaction with the BEC, which implies a superposition of signals with frequencies $\omega_1 = \omega_0\,[1+\beta\Delta(x,p;\varphi_n)]$ that contain terms $\beta\varphi_n$ and $\beta\varphi_n^2$, which is at the end the origin of the decoherence effect we illustrate in this work (see the details in the Appendix).

Replacing in Eq. (27) the summation over discrete levels of the BEC by integration as described in Sec. II (and changing the discrete subscript n to a continuous index φ) yields

$$\langle\hat{X}(t)\rangle_{\rm qcl} = \frac{1}{\sqrt{2\pi\,\Delta_\varphi^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\varphi\,dx\,dp \times \exp\left(-\varphi^2/2\Delta_\varphi^2\right) W(x,p)\,X_\varphi^{\rm cl}(x,p,t). \quad (33)$$

The integral in Eq. (33), with $W(x,p)$ and $X_\varphi^{\rm cl}(x,p,t)$ given by Eqs. (28) and (32), respectively, can be evaluated analytically, e.g., by formally introducing a Gaussian generating function $\mathcal{Z}(\mathbf{J})$,

$$\mathcal{Z}(\mathbf{J}) = e^{-C}\int\frac{d\mathbf{z}}{\pi\hbar\sqrt{2\pi\,\Delta_\varphi^2}}e^{-\mathbf{z}^T\cdot\mathsf{A}\cdot\mathbf{z}/2+(\mathbf{B}+\mathbf{J})^T\cdot\mathbf{z}} = \frac{2e^{-C}}{\hbar\Delta_\varphi\sqrt{\det\mathsf{A}}}\,e^{(\mathbf{B}+\mathbf{J})^T\cdot\mathsf{A}^{-1}\cdot(\mathbf{B}+\mathbf{J})/2}, \quad (34)$$

with $\mathbf{z} = (x,p,\varphi)$, $\mathbf{J} = (J_1,J_2,J_3)$,

$$\mathsf{A} = \begin{pmatrix} \frac{2m\omega_0}{\hbar} - \frac{3it\beta}{4m\omega_0} & 0 & -\frac{3it\beta}{4m^2\omega_0^3} \\ 0 & \frac{2}{m\hbar\omega_0} - \frac{3it\beta}{4m^3\omega_0^3} & 0 \\ -\frac{3it\beta}{4m^2\omega_0^3} & 0 & \frac{1}{\Delta_\varphi^2} - \frac{15it\beta}{4m^3\omega_0^5} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \frac{2m\omega_0x_0}{\hbar} \\ \frac{2p_0}{m\omega_0\hbar} \\ 0 \end{pmatrix}, \quad \text{and}$$

$$C = -i\omega_0 t + \frac{2}{\hbar\omega_0}\left(\frac{p_0^2}{2m} + \frac{1}{2}m\omega_0^2x_0^2\right). \quad (35)$$

Then defining $\overline{z_i}(t) \equiv \frac{\partial\mathcal{Z}}{\partial J_i}|_{\mathbf{J}=0}$, $i = 1,2,3$, we have

$$\langle\hat{X}(t)\rangle_{\rm qcl} = \frac{{\rm Re}[\overline{\varphi}(t)]}{m\omega_0^2} + {\rm Re}[\overline{x}(t)] + \frac{{\rm Im}[\overline{p}(t)]}{m\omega_0}, \quad (36)$$

from which an explicit analytical expression can be derived, although we will not cite it here.

Equations (33)–(36) are the main result of this section. In Fig. 3 we compare the analytical results in Eq. (36) with those obtained for $\langle\hat{X}(t)\rangle$ in Eq. (14) by numerical diagonalization of each $\hat{\mathcal{H}}_{\rm osc}(\varphi_n)$. At $t = 0$, the oscillator is prepared in a coherent state with $x_0 = 3$ a.u., and $p_0 = 0$, β has been set to 0.05 a.u. and $\omega_0 = 1.3$ a.u. which justifies the perturbative approach of Eqs. (29)–(32). The agreement between both results is good, and we proceed to use the quasiclassical equation (36) in order to characterize the decoherence in the short and the long time limits.

D. Time scale analysis

If the oscillator is not coupled to the condensate then it will show a coherent motion in which coherences will remain in time leading to recurrences in the oscillator dynamics. The action of the condensate quenches such recurrences. We shall assume that the coupling between the condensate and the oscillator is such that recurrences in the dynamics of the oscillator have been suppressed and therefore the decoherence

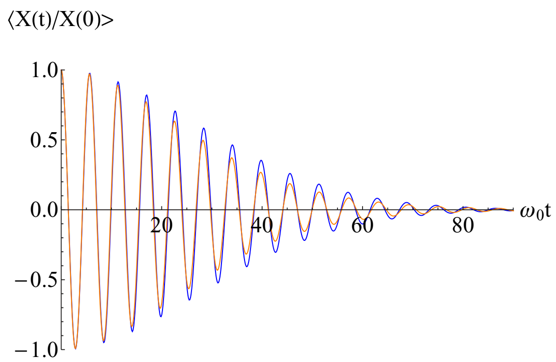

FIG. 3. (Color online) Comparison between the quasiclassical solution (36) (orange line) and the exact numerical solution of the equations of motion (blue line) in the perturbative regime ($\beta x_0^2/2m\omega_0^2 = 0.133$) for an initial coherent state with $x_0 = 3$, $p_0 = 0$, $\omega_0 = 1.3$, $\beta = 0.05$, and $\Delta_\varphi = 0.1$ (atomic units are used).

process occurs in a time scale much shorter than the dynamical recurrence time of the free (uncoupled) oscillator.

There are two relevant processes with regard to the time development and decay of $\langle \hat{X}(t)\rangle_{\rm qcl}$: the nonlinearity in the potential, and the interaction with the condensate. As already emphasized, when such nonlinearity does not exist the coupling between the oscillator and the condensate will not lead to a decay in oscillator's expectation values, even if variances and higher order fluctuations are affected by such coupling [23]. However, the nonlinear potential together with the oscillator-condensate interaction induces decoherence. A natural time scale linked to such nonlinearity can be defined as $t_\beta = (3\beta\hbar/4m^2\omega_0^2)^{-1}$. In addition, the interaction between the oscillator and the condensate introduces a different time scale, given by $t_\varphi = (3\beta\Delta_\varphi^2/4m^3\omega_0^5)^{-1}$. Both time scales are different and they are useful to understand the time development of oscillator's observables. We shall analyze the dynamics in two different situations, for t either smaller or larger than both characteristic time scales.

1. Case 1: $t \ll t_\beta, t_\varphi$

In this case the solution is accurately represented by

$$\langle \hat{X}(t)\rangle_{\rm qcl} \simeq e^{-t^2/2t_G^2}\left[x_0\cos(\omega_1 t) + \frac{p_0}{m\omega_0}\sin(\omega_1 t)\right], \quad (37)$$

with $\omega_1 = \omega_0 + \frac{3\beta\hbar}{4m^2\omega_0^2}\frac{\mathcal{H}(x_0,p_0;0)}{\hbar\omega_0} = \omega_0 + \Delta\omega_0$, and

$$t_G = t_\beta\left(\frac{\hbar\omega_0}{\mathcal{H}(x_0,p_0;0) + m\omega_0^2 x_0^2 t_\beta/t_\varphi}\right)^{1/2}. \quad (38)$$

It is apparent that t_G is a natural time scale associated with a Gaussian decoherence process taking place for times smaller than t_β and t_φ. Furthermore, a fully Gaussian decay, including a Gaussian tail, will develop if $t_G \ll t_\beta, t_\varphi$. That situation will appear in a fully semiclassical regime. In the quantum domain t_G will be of the order of or smaller than t_φ leading to a decay which will not be Gaussian. In the limit of harmonic potential a coherent motion as reported in [23] is recovered, emphasizing the relevance of the nonlinearity to the decoherence process.

2. Case 2: $t \gg t_\beta, t_\varphi$

In this case the solution can be represented by using a long time approximation by

$$\langle \hat{X}(t)\rangle_{\rm qcl} \simeq e^{-2\frac{\mathcal{H}(x_0,p_0;0)}{\hbar\omega_0}}\frac{m^{11/2}\omega_0^{13/2}}{\hbar^2\Delta\varphi}\frac{64}{9\sqrt{3}}(\beta t)^{-5/2} \times\left(x_0\cos\,\omega_0 t + \frac{p_0}{m\omega_0}\sin\,\omega_0 t\right) + O[(\beta t)^{-7/2}]. \quad (39)$$

The agreement of this expression with the numerical exact solutions to the quantum equations has been checked for small β ($\frac{\beta x_0^2}{\omega_0^2} \ll 1$). We observe that if time is much larger than t_β and t_φ the decay is algebraic. Notably the amplitude is exponentially damped as $\exp[-\mathcal{H}(x_0,p_0;0)/\hbar\omega_0]$, therefore this suggests that deeply within the quasiclassical regime, with consistency to the results of the previous section, it would be difficult to observe such power law decay, indicating that such dynamics would be observable mainly within the quantum domain.

The decay of observables shows an initial Gaussian decay for short times and a power law decay for longer times. Our results are consistent with those in [32–34] where Gaussian decoherence was studied.

In the next section we present numerical simulations as well as some analytical approximate expressions describing the dynamics of the system in the full quantum regime.

E. Quantum analysis

For the quantum case and an arbitrary value of β there are no analytical solutions. Numerical solutions can be obtained efficiently by diagonalizing $\hat{H}_{\rm osc} + \varphi\hat{X}$ over a truncated basis of stationary states of the harmonic oscillator centered at the origin, for each value of φ. Convergence with respect to the number of basis states used for the truncated diagonalization is checked for each particular value of β and for the initial state of the oscillator. These numerical results are used throughout the paper to compare with the different analytical approximations described.

In the general case, for arbitrary values of β, many different time-dependent terms will contribute to the sum in (14). However, in some limiting cases and under some approximations, analytical expressions can be obtained that describe the time decay with accuracy.

We will consider here as an illustration the case of an initial state of the oscillator involving only a few lower-energy states of the oscillator with frequency ω_0 and a value of $\hbar\beta \ll m^2\omega_0^3$.

Let us focus on the time-dependent part of Eq. (14),

$$\langle \hat{X}(t)\rangle - \langle \hat{X}\rangle^{\rm st} = \frac{1}{\sqrt{2\pi\,\Delta_\varphi^2}}\int_{-\infty}^{\infty} d\varphi\, e^{-\varphi^2/2\Delta_\varphi^2} \times\sum_{i<j} e^{-i(E_i^\varphi - E_j^\varphi)t/\hbar}F_{i,j}(\varphi) + \text{c.c.}, \quad (40)$$

where the superscript "st" stands for stationary and with $F_{i,j}(\varphi) \equiv \langle\psi_i^\varphi|\hat{\rho}_{\rm osc}(0)|\psi_j^\varphi\rangle\langle\psi_j^\varphi|\hat{X}|\psi_i^\varphi\rangle$.

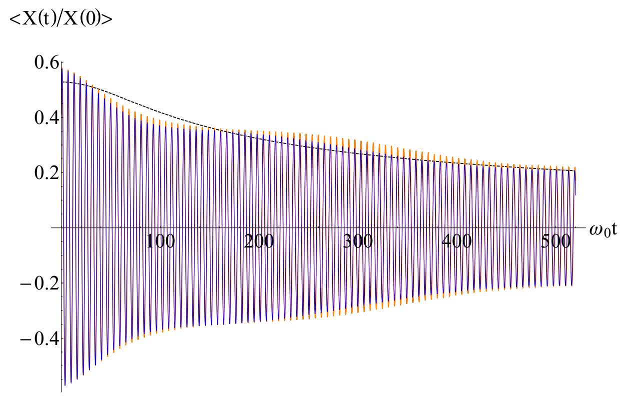

FIG. 4. (Color online) Mean value of position as a function of $\omega_0 t$ (normalized to its initial value) for an initial state of the oscillator $|\psi(0)\rangle = (1+i)\,|\psi_0\rangle/\sqrt{3} + i|\psi_1\rangle/\sqrt{3}$. Numerical result (solid line) and the analytical approximation given by Eq. (45) (dashed line) are almost indistinguishable in the figure. Frequency and amplitude of the signal are described with great accuracy by the analytical expression. The envelope of the analytical approximation in Eq. (46) is shown for reference (dotted line). The inset shows the details of the evolution for the expectation value of position in the large time region. $\beta = 0.05$, $\omega_0 = 1.3$, $\Delta_\varphi = 0.7$ (a.u.).

First, for not too large values of Δ_φ, it is enough to write the energy eigenvalues as a second order expansion in φ, $E_i^\varphi(\beta) = E_i^{\varphi=0}(\beta) + \gamma_i\varphi^2$ (the term linear in φ is zero because the anharmonicity is even in the x coordinate). A first order approximation in β for $E_i^{\varphi=0}(\beta)$ and γ_i,

$$E_i^{\varphi=0} = \hbar\omega_0\left(i+\frac{1}{2}\right) + \frac{3\hbar^2\beta\left(i^2+i+\frac{1}{2}\right)}{8m^2\omega_0^2}, \tag{41}$$

$$\gamma_i = -\frac{1}{2m\omega_0^2} + \frac{3\hbar\beta(2i+1)}{4m^3\omega_0^5}, \tag{42}$$

will be sufficient for the states that will contribute to the summation.

A valid approximation for $F_{i,j}(\varphi)$ can be obtained to zeroth order in β. Writing the initial state of the oscillator in terms of the stationary states of the harmonic oscillator $|\psi_n\rangle$ as $\hat{\rho}_{\rm osc}(0) = |\psi(0)\rangle\langle\psi(0)|$, with $|\psi(0)\rangle = \sum_n c_n|\psi_n\rangle$, and evaluating $\langle\psi_i^\varphi|\psi_n\rangle$ and $\langle\psi_j^\varphi|\hat{X}|\psi_i^\varphi\rangle$ as integrals over x, one obtains

$$\left\langle x\middle|\psi_n^\varphi\right\rangle = \sqrt{\frac{1}{2^n\, n!}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} H_n(y)e^{-y^2/2}, \tag{43}$$

with $y = \sqrt{\frac{m\omega_0}{\hbar}}(x - \frac{\varphi}{m\omega_0^2})$, for the displaced nth stationary state of the harmonic oscillator. Thus finally $F_{i,j}(\varphi)$ reads

$$F_{i,j}(\varphi) = G_{i,j}(\varphi)e^{-\varphi^2/(2m\hbar\omega_0^3)}, \tag{44}$$

where $G_{i,j}(\varphi) = g_{i,j}^{(0)} + g_{i,j}^{(2)}\varphi^2$ is a known polynomial of φ. The odd powers will not contribute to the integral. Furthermore, only terms up to the second order will be kept.

For an initial state being a combination of the first two lower-energy states of the harmonic oscillator, $|\psi(0)\rangle = c_0|\psi_0\rangle + c_1|\psi_1\rangle$, the mean value of position then reads

$$\begin{aligned}\langle\hat{X}(t)\rangle = \langle\hat{X}\rangle^{\rm st} + \sum_{n=0,1}\frac{e^{-i\omega_{n,n+1}t}}{\sqrt{2\Delta_\varphi^2}}\Bigg[g_{n,n+1}^{(0)}\sqrt{\frac{1}{1/\left(2\Delta_\varphi^2\right)+1/\left(2m\hbar\omega_0^3\right)+i(\gamma_n-\gamma_{n+1})t}}\\ +\frac{1}{2}g_{n,n+1}^{(2)}\left(\frac{1}{1/\left(2\Delta_\varphi^2\right)+1/\left(2m\hbar\omega_0^3\right)+i(\gamma_n-\gamma_{n+1})t}\right)^{3/2}\Bigg] + \text{c.c},\end{aligned} \tag{45}$$

where $\omega_{n,n+1} \equiv E_n^{\varphi=0} - E_{n+1}^{\varphi=0}$. Figure 4 shows exact numerical results compared to the analytical approximation given by Eq. (45). The inset presents the two curves for large time, where the analytical approximation is shown to reproduce both the frequency and the amplitude of the oscillations with great accuracy. The coefficients $g_{n,n+1}^{(0)}$ and $g_{n,n+1}^{(2)}$ depend on the coefficients in the Hermite polynomials, $H_n(x)$, as well as on c_0 and c_1 characterizing the state $|\psi(0)\rangle$,

$$g_{0,1}^{(0)} = \left(\frac{\hbar}{2m\omega_0}\right)^{1/2} c_0c_1^*, \quad g_{0,1}^{(2)} = -\left(\frac{2m\omega_0}{\hbar}\right)^{1/2}\frac{c_0^*c_1 + c_0c_1^*}{4m^2\omega_0^4},$$

$$g_{1,2}^{(0)} = 0, \quad g_{1,2}^{(2)} = \left(\frac{2m\omega_0}{\hbar}\right)^{1/2}\frac{c_0^*c_1 + 2c_0c_1^*}{4m^2\omega_0^4}.$$

If only the term $n=0$ is considered, a first order approximation in β is used for the difference $\gamma_0 - \gamma_1 = -3\hbar\beta/(2m^3\omega_0^5)$, and $G_{0,1}(\varphi)$ is taken as $g_{0,1}^{(0)}$. A good qualitative approximation is already obtained for this case,

$$\langle\hat{X}(t)\rangle \simeq \langle\hat{X}\rangle^{\rm st} + g_{0,1}^{(0)}e^{-i\omega_{0,1}t}\sqrt{\frac{1}{1+2\Delta_\varphi^2/\left(2m\hbar\omega_0^3\right)+3i\Delta_\varphi^2\frac{\hbar\beta}{m^3\omega_0^5}t}} + \text{c.c.} \tag{46}$$

The envelope of $\langle\hat{X}(t)\rangle$ in Eq. (46) is also shown for comparison in Fig. 4. Power law decay for the amplitude of the oscillations is observed, the different powers that contribute to the result depending on the initial state, being of the general form $t^{-k/2}$, with k being an integer.

IV. CONCLUSIONS

A detailed study of decoherence of an anharmonic oscillator in contact with a BEC trapped in a double-well potential is performed. The oscillator is coupled to the BEC through its position. In contrast with the harmonic oscillator case, for which coherent behavior has been reported, the anharmonic oscillator presents anomalous decoherence (nonexponential). In the quasiclassical domain there are two clearly distinguishable regimes. For short times, decoherence appears to be Gaussian with a well defined time scale. Such time scale depends on the degree of anharmonicity of the oscillator as well as on the energy distribution of the initial state of the whole system. The higher the anharmonicity of the oscillator and/or the energy distribution of the initial state, the faster the decay of coherence in short time scales. All that with consistency to a quantum-to-classical transition. On the other hand, at long times coherence decays algebraically. The particular power of the decay is characteristic of the initial states considered. The observation of both time regimes requires a very fine tuning of the initial state of the system, in particular of its initial energy distribution as measured by the energy variance. In the full quantum domain decoherence manifests itself in a combination of pure algebraic decay processes for all times of the form $t^{-k/2}$ (k an integer) according to the decomposition of the initial state of the oscillator in the harmonic oscillator number basis. This would allow one to observe coherent motion for longer times than what would be possible with an exponential decay. Our results show that a slight anharmonicity in the confining potential of the oscillator is sufficient to observe anomalous decoherence.

ACKNOWLEDGMENTS

We are grateful to Shmuel Gurvitz for useful discussions. D.A. acknowledges the warm hospitality of the Max Planck Institute for the Physics of Complex Systems at Dresden where part of this work was completed. Two of us (D.A. and S.B.) acknowledge financial support provided by Spanish MICINN (Grants No. FIS2010-19998 and FIS2013-41352) and the European Union (FEDER).

APPENDIX: THIRD HARMONIC CONTRIBUTION

It is known that the contribution of higher harmonics to the dynamics of the anharmonic oscillator may be relevant [31]. In fact, one can explicitly compute the solution $x(t)$ of $\ddot{x} = -\omega_0^2 x - \beta x^3$ with $x(0) = x_0$ and $\dot{x}(0) = 0$, which includes those higher harmonics, i.e.,

$$x(t) = x_0 \cos \omega t + \frac{\beta x_0^3}{32\omega^2}(\cos 3\omega t - \cos \omega t) + \frac{\beta^2 x_0^5}{1024\omega^4}(\cos 5\omega t - \cos \omega t) + \cdots,$$
$$\text{with} \quad \omega^2 = \frac{1}{16}\left[6\beta x_0^2 + 8\omega_0^2 + \sqrt{30\beta^2 x_0^4 + 96\beta x_0^2\omega_0^2 + 64\omega_0^4}\right]. \tag{A1}$$

Such solution is rather accurate even for a nonmoderate anharmonic contribution. However, in this work we restrict ourselves to the analysis of small anharmonicity for which $\frac{\beta x_0^2}{\omega_0^2} \ll 1$ and hence $\omega \approx \omega_0 + \frac{3\beta x_0^2}{8\omega_0}$. In such limit the phases are slightly corrected and the amplitudes of higher harmonics are negligible with respect to the first harmonic amplitude because $\frac{\beta x_0^2}{\omega_0^2} \ll 1$ implies $\frac{\beta x_0^2}{\omega^2} \ll 1$ and $\frac{\beta^2 x_0^4}{\omega^4} \ll 1$. So, in the limit we are considering, the main contribution to $x(t)$ comes from the first harmonic. Let us remark that as soon as the anharmonicity starts to be important higher harmonics should be considered but this is out of the aim of our present work.

At this stage a second aspect becomes relevant. The whole quantum signal (or its semiclassical approximation) is a superposition of individual $x(t;x_0,p_0,\varphi)$ that are averaged over initial conditions x_0,p_0 and over the parameter φ that measures the action of the condensate on the oscillator [see Eqs. (29)–(31)]. The phase correction depends on the initial condition and φ. Therefore, when averaging, the net result of such superposition will be a dephased signal, that eventually decays. It happens that the decay will be shown to be fast enough so that along the decay time, $x(t;x_0,p_0,\varphi)$, described by its first harmonic approximation, will be in fact an accurate description of the oscillators dynamics. The conclusion is that to characterize the decay in the parameter domain we are studying ($\frac{\beta x_0^2}{\omega_0^2} \ll 1$), it is enough to take into account the first correction in the phase and the first harmonic approximation.

[1] D. Hunger, S. Camerer, T. W. Hänsch, D. König, J. P. Kotthaus, J. Reichel, and P. Treutlein, Phys. Rev. Lett. **104**, 143002 (2010).

[2] P. Treutlein, C. Genes, K. Hammerer, M. Poggio, and P. Rabl, in *Cavity Optomechanics*, edited by M. Aspelmayer, T. Kippenberg, and F. Marquardt (Springer, Berlin, 2014), Chap. 14.

[3] J. Reichel, W. Hänsel, P. Hommelhoff, and T. W. Hänsch, Appl. Phys. B **72**, 81 (2001).

[4] K. C. Schwab and M. L. Roukes, Phys. Today **58**(7), 36 (2005).

[5] T. J. Kippenberg and K. J. Vahala, Science **321**, 1172 (2008).

[6] P. Treutlein, D. Hunger, S. Camerer, T. W. Hänsch, and J. Reichel, Phys. Rev. Lett. **99**, 140403 (2007).

[7] D. Meiser and P. Meystre, Phys. Rev. A **73**, 033417 (2006).

[8] C. Genes, D. Vitali, and P. Tombesi, Phys. Rev. A **77**, 050307 (2008).

[9] H. Ian, Z. R. Gong, Y.-x. Liu, C. P. Sun, and F. Nori, Phys. Rev. A **78**, 013824 (2008).

[10] S. Camerer, M. Korppi, A. Jöckel, D. Hunger, T. W. Hänsch, and P. Treutlein, Phys. Rev. Lett. **107**, 223001 (2011).
[11] B. Vogell, K. Stannigel, P. Zoller, K. Hammerer, M. T. Rakher, M. Korppi, A. Jöckel, and P. Treutlein, Phys. Rev. A **87**, 023816 (2013).
[12] K. Hammerer, M. Aspelmeyer, E. S. Polzik, and P. Zoller, Phys. Rev. Lett. **102**, 020501 (2009).
[13] W. K. Hensinger, D. W. Utami, H.-S. Goan, K. Schwab, C. Monroe, and G. J. Milburn, Phys. Rev. A **72**, 041405 (2005).
[14] S. Singh, M. Bhattacharya, O. Dutta, and P. Meystre, Phys. Rev. Lett. **101**, 263603 (2008).
[15] A. D. O'Connell, M. Hofheinz, M. Ansmann, R. C. Bialczak, M. Lenander, E. Lucero, M. Neeley, D. Sank, H. Wang, M. Weides, J. Wenner, J. M. Martinis, and A. N. Cleland, Nature (London) **464**, 697 (2010).
[16] M. P. Blencowe, J. Imbers, and A. D. Armour, New J. Phys. **7**, 236 (2005).
[17] S. A. Gurvitz and D. Mozyrsky, Phys. Rev. B **77**, 075325 (2008).
[18] D. Mozyrsky and I. Martin, Phys. Rev. Lett. **89**, 018301 (2002).
[19] C. A. Regal, J. D. Teufel, and K. W. Lehnert, Nat. Phys. **4**, 555 (2008).
[20] A. D. Armour and M. P. Blencowe, New J. Phys. **10**, 095004 (2008).
[21] M. P. Blencowe and A. D. Armour, New J. Phys. **10**, 095005 (2008).
[22] W. H. Zurek, Phys. Today **44**(10), 36 (1991).
[23] S. Brouard, D. Alonso, and D. Sokolovski, Phys. Rev. A **84**, 012114 (2011).
[24] D. Sokolovski and S. A. Gurvitz, Phys. Rev. A **79**, 032106 (2009).
[25] D. Sokolovski, Phys. Rev. Lett. **102**, 230405 (2009).
[26] E. Wigner, Phys. Rev. **40**, 749 (1932).
[27] H. Weyl, Z. Phys. **46**, 1 (1927).
[28] J. E. Moyal, Math. Proc. Cambridge Philos. Soc. **45**, 99 (1949).
[29] P. Gaspard, D. Alonso, and I. Burghardt, Adv. Chem. Phys. **90**, 105 (1995).
[30] J. Kevorkian and J. Cole, *Perturbation Methods in Applied Mathematics* (Springer-Verlag, New York, 1981).
[31] V. Marinca and N. Herisanu, *Nonlinear Dynamical Systems in Engineering: Some Approximate Approaches* (Springer, New York, 2012).
[32] D. Braun, F. Haake, and W. T. Strunz, Phys. Rev. Lett. **86**, 2913 (2001).
[33] W. T. Strunz, F. Haake, and D. Braun, Phys. Rev. A **67**, 022101 (2003).
[34] M. Žnidarič and T. Prosen, J. Opt. B **7**, 306 (2005).