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Preface

The basic reference for this course is [Walls and Milburn (2008)] from which the material of these notes has been extracted.

Chapter 1

Introduction

1.0.1 *Cuantización del campo electromagnético libre*

En estas secciones analizaremos algunas de las aplicaciones del formalismo de gases ideales cuánticos aplicadas a gases bosónicos. El campo electromagnético libre puede modelarse como un conjunto de osciladores armónicos desacoplados cuya energía puede ser cuantizada empleando las herramientas básicas que se estudian en un los cursos de Física Cuántica. La cuantización del campo electromagnético permite definir el concepto de *fotón*. Los fotones pueden ser tratados como partículas de un gas ideal cuántico con el formalismo que desarrollamos en la asignatura.

1.0.2 *Expresión clásica del campo electromagnético como un conjunto infinito de osciladores armónicos independientes*

Para una justificación rigurosa y asequible de los pasos dados en estas notas se puede consultar *Photons and Atoms: An Introduction to Quantum Electrodynamics* de Cohen Tannoudji [Cohen-Tannoudji *et al.* (1992)]. Las ecuaciones de Maxwell recogen la forma de las fuentes escalares y vectoriales tanto del campo eléctrico como del campo magnético. Junto con las relaciones de constitución nos dan toda la información necesaria sobre el campo electromagnético. En su forma más general, en medios materiales y en presencia tanto de carga libre (fuente de campo eléctrico) como de densidades de corriente (fuentes de campo magnético), tendremos

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon} \ ; \ \nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \ ; \ \nabla \times \mathbf{B} = \mu \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) \\
\mathbf{D} &= \varepsilon \mathbf{E} \ ; \ \mathbf{B} = \mu \mathbf{H}
\end{aligned}$$

Sin embargo consideraremos el vacío en ausencia de cargas y de densidades de corriente, es decir, la propagació n libre de radiación electromagnética sin fuentes de ningún tipo. En este caso, las ecuaciones de Maxwell y las relaciones de constitución adoptan una forma particularmente sencilla

$$\nabla \cdot \mathbf{D} = 0 \; ; \; \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \; ; \; \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$
$$\mathbf{D} = \varepsilon_0 \mathbf{E} \; ; \; \mathbf{B} = \mu_0 \mathbf{H}$$

En el tratamiento clásico, los campos $\mathbf{B}$ y $\mathbf{E}$ según las relaciones pueden obtenerse a partir de un par de potenciales $\mathbf{A}$ y ϕ

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

En caso de tratar en medios libres de carga, el potencial escalar se anula, permaneciendo sólo el potencial vector. Tenemos interés en expresar $\mathbf{A}$ en términos de un conjunto infinito de variables discretas para poder cuantizarlo posteriormente. Sabemos que tanto $\mathbf{H}$ como $\mathbf{E}$ pueden escribirse como solución de ecuaciones de onda de forma inmediata a partir de las ecuaciones de Maxwell. Un resultado que será de interés para nosotros es que el potencial vector $\mathbf{A}$ satisface igualmente una ecuación de ondas

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla (\nabla \cdot \mathbf{A}) - \mathit{\Delta} \mathbf{A} = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

El contraste de Coulomb impone que $\nabla \cdot \mathbf{A} = 0$ con lo que quedaría

$$\mathit{\Delta} \mathbf{A} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{1.1}$$

que es una ecuación de ondas tal. En lo sucesivo, supondremos que nuestro campo electromagnético se encuentra confinado en una región lo suficientemente grande para todos nuestros propósitos pero acotada. Tomaremos el sistema de unidades tal que el volumen de este recinto sea la unidad ($V = 1$). Podemos desarrollar entonces la solución de (1.1) en serie de Fourier en 3 dimensiones, quedando de la forma habitual

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r}) + \mathbf{a}_{\mathbf{k}}^* \exp(-i\mathbf{k} \cdot \mathbf{r})$$

Aquí, los $\mathbf{a}_{\mathbf{k}} \sim \exp(-i\omega t)$, como has comprobado en la sesión anterior. El potencial vector se representa como una superposición infinita de ondas planas monocromáticas (que individualmente son solución de la ecuación de ondas que satisface las condiciones de contorno). El principio de superposición nos garantiza que nuestra combinación es también solución de dicha ecuación (linealidad). La notación en forma de exponenciales simplifica mucho la manipulación de las expresiones pero se paga introduciendo los complejos conjugados $\mathbf{a}_{\mathbf{k}}^*$ en el desarrollo, lo garantiza que $\mathbf{A}(\mathbf{r}, t)$ sea una función real.

Se definen las variables canónicas en términos de las $\{\mathbf{a_k}, \mathbf{a_k^*}\}$. En función de estas nuevas variables el campo eléctrico y magnético en términos de un conjunto infinito pero discreto de variables. Conocidos los campos eléctrico y magnético, bastará con calcular la energía asociada al campo. Este será nuestro hamiltoniano clásico en la descripción de la radiación electromagnética.

$$\mathbf{q_k} = \frac{1}{\sqrt{\pi}}\left(\mathbf{a_k} + \mathbf{a_k^*}\right) \; ; \; \mathbf{p_k} = -\frac{i\omega}{\sqrt{\pi}}\left(\mathbf{a_k} - \mathbf{a_k^*}\right)$$

Las nuevas variables así definidas satisfacen las ecuaciones dinámicas

$$\frac{d}{dt}\left(\mathbf{q_k}\right) = \frac{-i\omega}{\sqrt{\pi}}\left(\mathbf{a_k} - \mathbf{a_k^*}\right) = \mathbf{p_k}$$

$$\frac{d}{dt}\left(\ \mathbf{p_k}\right) = \frac{\omega^2}{\sqrt{\pi}}\left(-\mathbf{a_k} - \mathbf{a_k^*}\right) = -\omega^2 \mathbf{q_k}$$

$$\text{es decir,} \quad \frac{d^2}{dt^2}\left(\mathbf{q_k}\right) = -\omega^2 \mathbf{q_k}$$

Se observa que tales ecuaciones son análogas a las variables de posición y momento para un sistema que oscila armónicamente (un *oscilador* para cada valor de $\mathbf{k}$). Podemos ahora invertir las definiciones de las coordenadas canónicas para expresar el potencial vector en estas nuevas variables. Se sigue

$$\mathbf{a_k} = \sqrt{\pi}\left(\mathbf{q_k} + \frac{i}{\omega}\mathbf{p_k}\right) \; ; \; \mathbf{a_k} = \sqrt{\pi}\left(\mathbf{q_k} - \frac{i}{\omega}\mathbf{p_k}\right)$$

de donde

$$\mathbf{A}\left(\mathbf{r}, t\right) = \sqrt{\pi}\sum_{\mathbf{k}}\left(\mathbf{q_k} + \frac{i}{\omega}\mathbf{p_k}\right)\exp\left(i\mathbf{kr}\right) + \left(\mathbf{q_k} - \frac{i}{\omega}\mathbf{p_k}\right)\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right).$$

Reexpresando los resultados se obtiene empleando la represetasión de Euler de las exponenciales complejas

$$\mathbf{A}\left(\mathbf{r}, t\right) = \sqrt{\pi}\sum_{\mathbf{k}} 2\mathbf{q_k}\left(\frac{\exp\left(i\mathbf{k}\cdot\mathbf{r}\right) + \exp\left(-i\mathbf{kr}\right)}{2}\right) - \frac{2}{\omega}\mathbf{p_k}\left(\frac{\exp\left(i\mathbf{k}\cdot\mathbf{r}\right) - \exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)}{2i}\right) =$$

$$= \sqrt{4\pi}\sum_{\mathbf{k}} \mathbf{q_k}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) - \frac{1}{\omega}\mathbf{p_k}\sin\left(\mathbf{k}\cdot\mathbf{r}\right).$$

De todo este análisis se tiene el potencial vector en términos de las variables canónicas de un conjunto de osciladores armónicos de frecuencia ω bien definida $\left(\omega = kc = k\left(\varepsilon_0\mu_0\right)^{-1/2}\right)$, calcularemos los campos eléctrico y magnético según las definiciones clásicas correspondientes:

$$\mathbf{E} = -\partial_t \mathbf{A} = \sqrt{4\pi}\sum_{\mathbf{k}} \mathbf{p_k}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) + \omega\mathbf{q_k}\sin\left(\mathbf{kr}\right)$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

En el caso del campo magnético el cálculo es ligeramente más tedioso. $\mathbf{p_k}$ y $\mathbf{q_k}$ no son funciones de la posición sino del tiempo, de modo que el rotacional actuará sólo sobre el coseno y el seno. Consideramos separadamente los dos sumandos

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ q_{kx}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) & q_{ky}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) & q_{kz}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) \end{vmatrix} =$$

$$= \left\{ \begin{array}{c} \mathbf{i}\left(\partial_y q_{kz}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) - \partial_z q_{ky}\cos\left(\mathbf{k}\cdot\mathbf{r}\right)\right) + \\ +\mathbf{j}\left(\partial_z q_{kx}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) - \partial_x q_{kz}\cos\left(\mathbf{k}\cdot\mathbf{r}\right)\right) + \\ +\mathbf{k}\left(\partial_x q_{ky}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) - \partial_y q_{kx}\cos\left(\mathbf{k}\cdot\mathbf{r}\right)\right) \end{array} \right\} =$$

$$= \left\{ \begin{array}{c} \mathbf{i}\left(-k_y q_{kz}\sin\left(\mathbf{k}\cdot\mathbf{r}\right) + k_z q_{ky}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\right) + \\ +\mathbf{j}\left(-k_z q_{kx}\sin\left(\mathbf{k}\cdot\mathbf{r}\right) + k_x q_{kz}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\right) + \\ +\mathbf{k}\left(-k_x q_{ky}\sin\left(\mathbf{k}\cdot\mathbf{r}\right) + k_y q_{kx}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\right) \end{array} \right\} =$$

$$= \mathbf{q_k} \times \mathbf{k}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)$$

El segundo sumando es totalmente análogo sólo que contiene un coseno y conlleva por lo tanto un cambio de signo respecto del primero. El campo magnético quedaría finalmente como

$$\mathbf{B} = \sqrt{4\pi}\sum_{\mathbf{k}} \mathbf{q_k} \times \mathbf{k}\sin\left(\mathbf{k}\cdot\mathbf{r}\right) + \frac{1}{\omega}\mathbf{p_k} \times \mathbf{k}\cos\left(\mathbf{k}\cdot\mathbf{r}\right)$$

La expresión para la densidad de energía es la usual en términos de la suma de los cuadrados de los módulos de los campos eléctrico y magnético. Sabemos del Electromagnetismo Clásico, que la energía electromagnética total en un volumen (si el medio es lineal) se puede expresar en términos de los vectores $\mathbf{E}, \mathbf{D}, \mathbf{B}$ y $\mathbf{H}$ como

$$U = \frac{1}{2}\int_V dV\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right) = \frac{1}{2}\int_V dV\left(\varepsilon_0\mathbf{E}^2 + \mu_0\mathbf{B}^2\right)$$

En el sistema *cgs*, muy utilizado en Electromagnetismo, esta expresión adopta la forma

$$U = \frac{1}{8\pi}\int_V dV\left(\mathbf{E}^2 + \mathbf{B}^2\right).$$

Lo realmente importante de la expresió es que aparece una suma de campo eléctrico y magnético al cuadrado. A la hora de evaluarlos, tendremos en cuenta la ortogonalidad de las funciones sin y cos dentro del volumen en que hemos efectuado el desarrollo en serie de Fourier. Se anulan los todos los sumandos con la siguiente estructura

$$\int_V dV\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\sin\left(\mathbf{k}'\mathbf{r}\right) = 0\ ; \quad \int_V dV\cos\left(\mathbf{k}\cdot\mathbf{r}\right)\cos\left(\mathbf{k}'\cdot\mathbf{r}\right) = 0$$

$$\int_V dV\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\cos\left(\mathbf{k}'\mathbf{r}\right) = 0\ ; \quad \int_V dV\cos\left(\mathbf{k}\cdot\mathbf{r}\right)\sin\left(\mathbf{k}'\cdot\mathbf{r}\right) = 0$$

$$\int_V dV\cos\left(\mathbf{k}\cdot\mathbf{r}\right)\sin\left(\mathbf{k}\cdot\mathbf{r}\right) = 0\ ;$$

$$\text{con } \mathbf{k} \neq \mathbf{k}'$$

La energía total asociada al campo eléctrico (escrita en *cgs*) queda como

$$\begin{aligned} U_e &= \frac{1}{8\pi}\int_V dV\mathbf{E}^2 \\ &= \frac{4\pi}{8\pi}\sum_{\mathbf{k},\mathbf{k}'}\int_V dV\left(\mathbf{p_k}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) + \omega\mathbf{q_k}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\right)\left(\mathbf{p_{k'}}\cos\left(\mathbf{k}'\cdot\mathbf{r}\right) + \omega\mathbf{q_{k'}}\sin\left(\mathbf{k}'\cdot\mathbf{r}\right)\right) = \\ &= \frac{1}{2}\sum_{\mathbf{k}}\int_V dV\left(\mathbf{p}_\mathbf{k}^2\cos^2\left(\mathbf{k}\cdot\mathbf{r}\right) + \omega^2\mathbf{q}_\mathbf{k}^2\sin^2\left(\mathbf{k}\cdot\mathbf{r}\right)\right). \end{aligned}$$

Por otra parte, la contribución del campo magnético es

$$U_m = \frac{1}{8\pi}\int_V dV\mathbf{B}^2 = \frac{1}{2}\sum_{\mathbf{k}}\int_V dV\left(\mathbf{q_k}\times\mathbf{k}\right)^2\sin^2\left(\mathbf{k}\cdot\mathbf{r}\right) + \frac{1}{\omega^2}\left(\mathbf{p_k}\times\mathbf{k}\right)^2\cos^2\left(\mathbf{k}\cdot\mathbf{r}\right).$$

Nótese que el contraste (*gauge*) de Coulomb $\mathbf{k}\perp\mathbf{A}$.

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \nabla\cdot\left(\sum_{\mathbf{k}}\mathbf{q_k}\cos\left(\mathbf{k}\cdot\mathbf{r}\right) - \frac{1}{\omega}\mathbf{p_k}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\right) = 0 \\
\nabla\cdot\mathbf{A} &= -\sum_{\mathbf{k}}\left(k_x q_{kx} + k_y q_{ky} + k_z q_{kz}\right)\sin\left(\mathbf{k}\cdot\mathbf{r}\right) \\
&\quad - \frac{1}{\omega}\sum_{\mathbf{k}}\left(k_x p_{kx} + k_y p_{ky} + k_z p_{kz}\right)\cos\left(\mathbf{k}\cdot\mathbf{r}\right) = 0
\end{aligned}$$

Dado que la igualdad debe darse para todo$\mathbf{r}$, será preciso exigir que $\mathbf{q_k}\mathbf{k} = \mathbf{p_k}\mathbf{k} = 0$, es decir, el vector de onda $\mathbf{k}$ de la radiación es perpendicular al plano que contiene a las variables canónicas $\mathbf{q_k}$ y $\mathbf{p_k}$ que darán cuenta de la *polarización* de la radiación. Por este motivo, podemos expresar los productos vectoriales $\left(\mathbf{q_k}\times\mathbf{k}\right)^2$ y $\left(\mathbf{p_k}\times\mathbf{k}\right)^2$ como

$$\left(\mathbf{q_k}\times\mathbf{k}\right)^2 = \mathbf{q_k^2}\mathbf{k}^2\ ;\ \ \left(\mathbf{p_k}\times\mathbf{k}\right)^2 = \mathbf{p_k^2}\mathbf{k}^2$$

lo que nos lleva a

$$U_m = \frac{1}{8\pi}\int_V dVB^2 = \frac{1}{2}\sum_{\mathbf{k}}\int_V dV\left(\mathbf{q_k}\mathbf{k}\right)^2\sin^2\left(\mathbf{kr}\right) + \frac{1}{\omega^2}\left(\mathbf{p_k}\mathbf{k}\right)^2\cos^2\left(\mathbf{k}\cdot\mathbf{r}\right).$$

Si tomamos $c = 1$, podemos expresar la relación de dispersión como $k = \omega/c = \omega$ se sigue

$$U_m = \frac{1}{8\pi}\int_V dVB^2 = \frac{1}{2}\sum_{\mathbf{k}}\int_V dV\left(\mathbf{q_k^2}\omega^2\sin^2\left(\mathbf{k}\cdot\mathbf{r}\right) + \mathbf{p_k^2}\cos^2\left(\mathbf{k}\cdot\mathbf{r}\right)\right)$$

Es decir, $U_m = U_e$. La suma de ambos, nos dará

$$\begin{aligned}
\mathcal{H} &= \sum_{\mathbf{k}}\int_V dV\left(\mathbf{q_k^2}\omega^2\sin^2\left(\mathbf{k}\cdot\mathbf{r}\right) + \mathbf{p_k^2}\cos^2\left(\mathbf{k}\cdot\mathbf{r}\right)\right) = \\
&= \frac{1}{2}\sum_{\mathbf{k}}\left(\mathbf{q_k^2}\omega^2 + \mathbf{p_k^2}\right).
\end{aligned}$$

Este último paso en que hemos evaluado las integrales, depende de cómo se tomen las funciones ortogonales en las que expandimos el campo $\mathbf{A}$ al principio del desarrollo (los senos y cosenos), es decir, depende de cómo se definan las $\mathbf{a_k}$ de las cuales no dijimos nada en su momento salvo cómo era su dependencia temporal. Pueden tomarse normalizadas a la unidad o a cualquier otro valor que nos interese. En este caso, vemos que conviene tomarlas *normalizadas* al valor $1/2$ para reproducir en el hamiltoniano, la suma de infinitos osciladores armónicos. Otra modificación que podemos introducir en la notación tiene que ver con la ortogonalidad de las variables canónicas y el vector de onda $\mathbf{k}$ de la radiación. Esto hace que $\mathbf{q_k}$ y $\mathbf{p_k}$

estén contenidos en un plano y que sólo tengan dos componentes en un sistema cartesiano en que tomemos uno de los ejes según la dirección de propagación de la onda (digamos el eje z). Podemos separar las componentes de forma explícita convirtiendo el sumatorio en un doble sumatorio e introduciendo una nueva variable muda α

$$\mathcal{H} = \sum_{\mathbf{k},\alpha} \frac{1}{2}\left(q_{\mathbf{k},\alpha}^2\omega^2 + p_{\mathbf{k},\alpha}^2\right) = \sum_{\mathbf{k},\alpha} \mathcal{H}_{\mathbf{k},\alpha}$$

α puede valer 1 o 2 (haciendo referencia a las componentes no nulas de los vectores $\mathbf{q_k}$ y $\mathbf{p_k}$ en el sistema cartesiano que hemos elegido). Como vemos, la energía de cada *oscilador* viene caracterizado por el vector de onda asociado y su polarización (dada por el parámetro α).

1.0.3 *Cuantización de las expresiones clásicas*

Por el momento se han expresado los campos $\mathbf{A}, \mathbf{E}, \mathbf{B}$ y $\mathcal{H}$ en términos de unas variables canónicas que obedecen la ecuaciónes de osciladores armónicos. Vamos a definir ahora los observables $Q_{\mathbf{k}\alpha}$ y $P_{\mathbf{k},\alpha}$ correspondientes a través de sus relaciones de conmutación y sus ecuaciones de valores propios.[1]

$$[Q_{\mathbf{k}\alpha}, P_{\mathbf{k}',\alpha'}] = i\hbar\delta_{\mathbf{kk}'}\delta_{\alpha\alpha'} = ...\,(\hbar = 1)\,... = i\delta_{\mathbf{kk}'}\delta_{\alpha\alpha'}$$
$$Q_{\mathbf{k}\alpha}\,|q_{\mathbf{k}\alpha}\rangle = q_{\mathbf{k}\alpha}\,|q_{\mathbf{k}\alpha}\rangle \;\; ; \;\; P_{\mathbf{k}\alpha}\,|p_{\mathbf{k}\alpha}\rangle = p_{\mathbf{k}\alpha}\,|p_{\mathbf{k}\alpha}\rangle$$
$$\{|q_{\mathbf{k}\alpha}\rangle\} \;\text{ y }\; \{|p_{\mathbf{k}\alpha}\rangle\} \;\text{ son bases mixtas de }\varepsilon$$

Hemos vuelto a hacer un cambio de unidades para que $\hbar = 1$ compactando así la notación y suponiendo todas las precauciones y consideraciones pertinentes. Hemos introducido también las bases mixtas (continuas en el índice $\mathbf{k}$ y discretas en el índice α) del espacio de los estados total $\varepsilon = \varepsilon_1 \otimes \varepsilon_2 \otimes ...$ donde los ε_i son los espacios de los estados de los osciladores individuales. Dado que $Q_{\mathbf{k}\alpha}$ y $P_{\mathbf{k},\alpha}$ son operadores hermíticos, el análogo cuántico del potencial magnético $\mathbf{A}$, el campo eléctrico $\mathbf{E}$ y el campo magnético $\mathbf{B}$, también lo serán (ya que estos campos son función de $Q_{\mathbf{k}\alpha}$ y $P_{\mathbf{k}\alpha}$). Dado que trabajamos con sistemas que se rigen por el hamiltoniano de osciladores armónicos, resulta muy útil definir análogamente a como se hace en el estudio de un oscilador armónico cuántico *real* unos operadores de creación y destrucción en términos de los observables $Q_{\mathbf{k}\alpha}$ y $P_{\mathbf{k}\alpha}$.

$$c_{\mathbf{k}\alpha} = \frac{1}{\sqrt{2\omega}}\left(\omega Q_{\mathbf{k}\alpha} + iP_{\mathbf{k}\alpha}\right) \;\; ; \;\; c_{\mathbf{k}\alpha}^\dagger = \frac{1}{\sqrt{2\omega}}\left(\omega Q_{\mathbf{k}\alpha} - iP_{\mathbf{k}\alpha}\right)$$

[1] Estamos diciendo sin más que $Q_{\mathbf{k}\alpha}$ y $P_{\mathbf{k}\alpha}$ son observables y escribiendo las bases correspondientes formadas por sus vectores propios. La justificación intuitiva para esto descansa en la estructura del hamiltoniano, formalmente idéntico al de un conjunto de osciladores reales. Hacemos entonces la analogía entre las variables canónicas $p_{\mathbf{k}\alpha}$ y $q_{\mathbf{k}\alpha}$ (sin tener demasiado claro lo que significan en realidad físicamente) y las variables de posición y momento $\mathbf{r}$ y $\mathbf{p}$ que ya conocemos sobradamente. Los análogos cuánticos de estas últimas sí que son observables, lo que nos da cierto derecho a exigirlo para los operadores que acabamos de definir.

La forma que escogemos para escribir $\mathbf{A}, \mathbf{E}$ y $\mathbf{B}$ es la siguiente

$$\mathbf{A} = \sum_{\mathbf{k},\alpha} c_{\mathbf{k}\alpha}\mathbf{A}_{\mathbf{k}\alpha} + c^{\dagger}_{\mathbf{k}\alpha}\mathbf{A}^{*}_{\mathbf{k}\alpha} \; ; \; \mathbf{E} = \sum_{\mathbf{k},\alpha} c_{\mathbf{k}\alpha}\mathbf{E}_{\mathbf{k}\alpha} + c^{\dagger}_{\mathbf{k}\alpha}\mathbf{E}^{*}_{\mathbf{k}\alpha} \; ; \; \mathbf{B} = \sum_{\mathbf{k},\alpha} c_{\mathbf{k}\alpha}\mathbf{B}_{\mathbf{k}\alpha} + c^{\dagger}_{\mathbf{k}\alpha}\mathbf{B}^{*}_{\mathbf{k}\alpha}$$

Comparando $\mathbf{A}$ con su análogo cuántico, vemos que los $A_{\mathbf{k}\alpha}$ y los $A^{*}_{\mathbf{k}\alpha}$ deben tener la forma

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \sqrt{\pi}\sum_{\mathbf{k}} \left(\mathbf{q}_{\mathbf{k}} + \frac{i}{\omega}\mathbf{p}_{\mathbf{k}}\right)\exp(i\mathbf{k}\mathbf{r}) + \left(\mathbf{q}_{\mathbf{k}} - \frac{i}{\omega}\mathbf{p}_{\mathbf{k}}\right)\exp(-i\mathbf{k}\cdot\mathbf{r}) \Rightarrow \\ &\Rightarrow \mathbf{A}_{\mathbf{k}\alpha} = \sqrt{\frac{2\pi}{\omega}}\exp(i\mathbf{k}\cdot\mathbf{r})\,\hat{u}_{\mathbf{k}\alpha} \end{aligned}$$

Para $\mathbf{E}$ tendríamos

$$\begin{aligned} \mathbf{E} &= \sqrt{\pi}\sum_{\mathbf{k}} \mathbf{p}_{\mathbf{k}}\left(\exp(i\mathbf{k}\cdot\mathbf{r}) + \exp(-i\mathbf{k}\cdot\mathbf{r})\right) - i\omega\mathbf{q}_{\mathbf{k}}\left(\exp(i\mathbf{k}\cdot\mathbf{r}) - \exp(-i\mathbf{k}\cdot\mathbf{r})\right) = \\ &= \sqrt{\pi}\sum_{\mathbf{k}} (\mathbf{p}_{\mathbf{k}} - i\omega\mathbf{q}_{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r}) + (\mathbf{p}_{\mathbf{k}} + i\omega\mathbf{q}_{\mathbf{k}})\exp(-i\mathbf{k}\cdot\mathbf{r}) \\ \mathbf{E} &= \sum_{\mathbf{k},\alpha} c_{\mathbf{k}\alpha}\left(-\sqrt{\frac{2\pi}{\omega}}i\exp(i\mathbf{k}\cdot\mathbf{r})\,\hat{u}_{\mathbf{k}\alpha}\right) + c^{\dagger}_{\mathbf{k}\alpha}\left(\sqrt{\frac{2\pi}{\omega}}i\exp(-i\mathbf{k}\cdot\mathbf{r})\,\hat{u}_{\mathbf{k}\alpha}\right) \Rightarrow \\ &\Rightarrow \mathbf{E}_{\mathbf{k}\alpha} = -\sqrt{\frac{2\pi}{\omega}}i\exp(i\mathbf{k}\cdot\mathbf{r})\,\hat{u}_{\mathbf{k}\alpha} \end{aligned}$$

Por último, dado que $\mathbf{E}$ y $\mathbf{B}$ son ortogonales, tendremos $\mathbf{B}_{\mathbf{k}\alpha} = \hat{n} \times \mathbf{E}_{\mathbf{k}\alpha}$ donde $\hat{n} = \mathbf{k}/k$. El procedimiento es exactamente el mismo que en el caso clásico. Notemos que los vectores que acompañan a los operadores de creación y destrucción verifican ciertas relaciones (triviales dada su forma) que son útiles a la hora de calcular el hamiltoniano cuántico H.

$$\int d^3r\,\mathbf{A}_{\mathbf{k}\alpha}\mathbf{A}^{*}_{\mathbf{k}\alpha} = \frac{2\pi}{\omega}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\alpha'}$$

$$\int d^3r\left(\mathbf{E}_{\mathbf{k}\alpha}\mathbf{E}^{*}_{\mathbf{k}\alpha} + \mathbf{B}_{\mathbf{k}\alpha}\mathbf{B}^{*}_{\mathbf{k}\alpha}\right) = 4\pi\omega\delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\alpha'}$$

Aplicando este resultado, vemos que el hamiltoniano queda como

$$H = \frac{1}{8\pi}\sum_{\mathbf{k}\alpha} 4\pi\omega\left(\mathbf{E}_{\mathbf{k}\alpha}\mathbf{E}^{*}_{\mathbf{k}\alpha} + \mathbf{B}_{\mathbf{k}\alpha}\mathbf{B}^{*}_{\mathbf{k}\alpha}\right) = \sum_{\mathbf{k}\alpha}\frac{\omega}{2}\left(c_{\mathbf{k}\alpha}c^{\dagger}_{\mathbf{k}\alpha} + c^{\dagger}_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha}\right)$$

Este resultado era totalmente esperable, dado que corresponde a la suma de los hamiltonianos de infinitos osciladores armónicos cuánticos. Uno para cada polarización α y para cada valor de $\mathbf{k}$, es decir, para cada frecuencia bien definida. La energía total de un sistema de este tipo, viene dada por

$$E = \sum_{\mathbf{k}\alpha}\left(N_{\mathbf{k}\alpha} + \frac{1}{2}\right)\omega$$

Su momento también puede definirse de forma precisa. Existe un problema derivada del hecho de que los osciladores tienen energía no nula aún en el nivel fundamental.

Para evitar esto, podríamos tratar de introducir un cambio en la *referencia* de energías. Lo haremos aprovechando las relaciones de conmutación de los operadores de creación y destrucción.

$$\begin{aligned}\left[c_{\mathbf{k}\alpha}, c^{\dagger}_{\mathbf{k}\alpha}\right] &= \frac{1}{2\omega}\Big(\left(\omega Q_{\mathbf{k}\alpha} + iP_{\mathbf{k}\alpha}\right)\left(\omega Q_{\mathbf{k}\alpha} - iP_{\mathbf{k}\alpha}\right) - \left(\omega Q_{\mathbf{k}\alpha} - iP_{\mathbf{k}\alpha}\right)\left(\omega Q_{\mathbf{k}\alpha} + iP_{\mathbf{k}\alpha}\right) \Big) \\ &= i\left[P_{\mathbf{k}\alpha}, Q_{\mathbf{k}\alpha}\right] = 1 \end{aligned} \tag{1.2}$$

Haciendo uso de esta relación, escribimos

$$\begin{aligned} H &= \frac{\omega}{2}\sum_{\mathbf{k}\alpha}\left(1 - c_{\mathbf{k}\alpha}c^{\dagger}_{\mathbf{k}\alpha} + c^{\dagger}_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha} + c_{\mathbf{k}\alpha}c^{\dagger}_{\mathbf{k}\alpha} + c^{\dagger}_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha}\right) = \\ &= \frac{\omega}{2}\sum_{\mathbf{k}\alpha}\left(1 + 2c^{\dagger}_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha}\right) = \frac{1}{2}\sum_{\mathbf{k}\alpha}\omega + \sum_{\mathbf{k}\alpha}\omega c^{\dagger}_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha} \end{aligned}$$

El primer sumatorio es formalmente infinito, no obstante se considerarán diferencias de energía por lo que se puede considerar el hamiltoniano

$$H = \sum_{\mathbf{k}\alpha}\omega c^{\dagger}_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha}$$

La energía en este caso será

$$E = \sum_{\mathbf{k}\alpha} N_{\mathbf{k}\alpha}\omega$$

El momento del sistema de osciladores puede definirse como

$$\mathbf{P} = \sum_{\mathbf{k}\alpha} N_{\mathbf{k}\alpha}\mathbf{k}$$

Chapter 2

States of the EM-field

2.1 Fock states

When quantizing the Electromagnetic field it was introduced the Fock representation in which it is specified the number of photons of the field, regardless its polarizaion, with a given wavevector $\mathbf{k}$ present in the field. Such number of photons is denoted by $n_{\mathbf{k}}$ and its corresponding Fock vector is obtained from the vacumm state $|0\rangle$ (no photons in the field) by the consecutive application of the creation operator of the field $a_{\mathbf{k}}^{\dagger}$ at wavevector $\mathbf{k}$, *i.e.*

$$|n_{\mathbf{k}}\rangle = \frac{(a_{\mathbf{k}})^{n_{\mathbf{k}}}}{\sqrt{n_{\mathbf{k}}!}}|0\rangle. \tag{2.1}$$

Fock states form an orthogonal basis with a resolution of identity for each mode $\mathbf{k}$

$$\mathbb{1} = \sum_{\mathbf{k}} |n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|. \tag{2.2}$$

When several modes with wavevectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \cdots$ the Fock state of the multimode field is constructed as a tensor product of single mode states as

$$|\mathbf{n}\rangle =: |n_{\mathbf{k}_1}\rangle \otimes |n_{\mathbf{k}_2}\rangle \otimes |n_{\mathbf{k}_3}\rangle \otimes \cdots = |n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \cdots\rangle, \tag{2.3}$$

with $\langle \mathbf{n}|\mathbf{m}\rangle = \delta_{\mathbf{nm}}$ and

$$\sum_{n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \cdots} |n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3}\rangle\langle n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3}| = \mathbb{1}. \tag{2.4}$$

2.2 Coherent states

Another important set of states of the EM-field are *coherent states*. They minimize the uncentainty relation in both amplitude and phase. To keep the discussion simple we shall treat the single mode case. Let us proceed first with some definitions to continue after with an important set of properties that coherent states may exhibit [Scully and Zubairy (1997)].

Note: Maybe we can introduce this concept as an holomorphic representation for the harmonic oscilator "'algebra"'.

2.2.1 *Displacement operator*

The discplacement operator $D(\alpha)$ is define as

$$D(\alpha) =: e^{\alpha a^\dagger - \alpha^* a}, \quad \alpha \in \mathbb{C}. \tag{2.5}$$

Form this definition, $[a, a^\dagger] = 1$ and the relation

$$\begin{aligned} Z(X,Y) &= \log(\exp X \exp Y) \\ &= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] \\ &\quad - \frac{1}{24}[Y,[X,[X,Y]]] \\ &\quad - \frac{1}{720}([[[[X,Y],Y],Y],Y] + [[[[Y,X],X],X],X]) \\ &\quad + \frac{1}{360}([[[[X,Y],Y],Y],X] + [[[[Y,X],X],X],Y]) \\ &\quad + \frac{1}{120}([[[[Y,X],Y],X],Y] + [[[[X,Y],X],Y],X]) + \cdots \end{aligned} \tag{2.6}$$

follows

$$\begin{aligned} D(\alpha) &= e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2} \\ D^\dagger(\alpha) &= D^{-1}(\alpha) = D(-\alpha). \end{aligned} \tag{2.7}$$

The equalities shown below justify the name *displacement* to the $D(\alpha)$ operators, because

$$\begin{aligned} D^\dagger(\alpha) a D(\alpha) &= a + \alpha \\ D^\dagger(\alpha) a^\dagger D(\alpha) &= a^\dagger + \alpha^*. \end{aligned}$$

Both equations can be proven from the Hadamard lema for two operators X and Y. (✍ **Exercise** 2.2.1)

$$e^Y X e^{-Y} = X + [Y,X] + \frac{1}{2!}[Y,[Y,X]] + \frac{1}{3!}[Y,[Y,[Y,X]]] + \cdots. \tag{2.8}$$

2.2.2 *Definition*

A coherent state can be defined as the application of the displacement operator on vacuum, *i.e.*

$$|\alpha\rangle =: D(\alpha)|0\rangle. \tag{2.9}$$

This state has a number of properties that we summarize in the following. The interested reader should prove them.

(1) A coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

(✍ **Exercise** 2.2.2)

Note that a is not hermitian therefore α belongs to the field of complex number.

(2) Coherent states has an expansion in Fock states of the form

$$|\alpha\rangle = e^{|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

(✍ **Exercise** 2.2.3)

(3) Coherent states are normalized. (✍ **Exercise** 2.2.4)

(4) A coherent state does not have a definite number of photons. In fact one can compute the probability that a measurement of the number operator $N =: a^\dagger a$ results in a particular value n. Such probability is is given

$$P_{|\alpha\rangle}(n) =: P(n) = \left|\langle n|\alpha\rangle\right|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}. \quad \text{(Poisson distribution)} \tag{2.10}$$

Therefore a set of measurements of N are distributed with a mean number of photons $\langle n \rangle$ and variance

$$\langle n \rangle = \bar{n} = \overbrace{(\Delta n)^2}^{variance} = |\alpha|^2. \tag{2.11}$$

(✍ **Exercise** 2.2.5)

(5) Coherent states are not orthogonal. In fact

$$\langle \alpha | \beta \rangle = e^{-|\alpha - \beta|^2}. \tag{2.12}$$

(6) Coherent states form an over complete basis. Therefore they define a resolution of identity and furthermore any coherent state can be written as an expansion over coherent states. In particular

$$\begin{aligned} \mathbb{1} &= \frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha| := \frac{1}{\pi} \int d(\mathrm{Re}\alpha) d(\mathrm{Im}\alpha) \, |\alpha\rangle\langle\alpha| \\ |\alpha\rangle &= \frac{1}{\pi} \int d^2\beta \, \langle \beta|\alpha\rangle \, |\beta\rangle \quad \text{(over completeness)}. \end{aligned} \tag{2.13}$$

2.2.2.1 *Coherent states are almost classical states*

One of the interesting features of coherent states is their close relation to classical behavior. To illustrate this fact we consider a harmonic oscillator initially in a coherent state $|\alpha\rangle$ that undergoes a unitary evolution up to time t. At that precise instant it is measured the position X of the oscillator. We may ask for the probability that in such measurement the particle is found at position x, *i.e.* $p_{|\alpha\rangle}(x) =: p(x)$ which is given by

$$\begin{aligned} p(x) &= \left|\langle x|e^{-i(\omega(a^\dagger a)+\frac{1}{2})t}|\alpha\rangle\right|^2 \\ &= e^{-|\alpha|^2 - i\omega t}\frac{1}{y\sqrt{\pi}}\left|\sum_{n=0}^{\infty}\frac{(e^{-i(\omega t-\phi n)}|\alpha|/2)^n}{\sqrt{n!}}H_n(x/y)\right|^2, \end{aligned} \tag{2.14}$$

where $y = \sqrt{\hbar/m\omega}$, $\alpha = |\alpha|e^{i\phi}$ and $H_n(x)$ is the Hermite polynomial of order n. After some manipulations it is obtained

$$p(x) = e^{-|\alpha|^2-(x/y)^2}\frac{1}{y\sqrt{\pi}}e^{-2|\alpha|^2\cos^2(\omega t-\phi)}e^{2\sqrt{2}\cos(\omega t-\phi)/y}. \tag{2.15}$$

In addition the position of a classical oscillator at time t turn out to be $x_c(t) = \sqrt{2}|\alpha|\cos(\omega t - \phi)$ and therefore

$$p(x) = \frac{1}{\sigma_x\sqrt{2\pi}}e^{-(x-x_c(t))^2/2\sigma_x^2} \text{ with } \quad \sigma_x = y/\sqrt{2}. \tag{2.16}$$

Thus $p(x)$ is a Gaussian probability distribution with a time independent variance σ_x and it is centered at the classical trajectory of the classical counterpart of the quantum system. The corresponding momentum distribution $p(p)$ is also Gaussian (✍ **Exercise** 2.2.6) and will lead to

$$\sigma_x\sigma_p = \frac{\hbar}{2}. \tag{2.17}$$

☞Notice that in general a quantum state satisfies $\sigma_x\sigma_p \geq \frac{\hbar}{2}$.

which expresses that coherent states have minimum uncertainty.

Exercises

Exercise 2.2.1. Prove the Hadamard lema.

Exercise 2.2.2. Prove equation (2.10)

Exercise 2.2.3. Prove equation (2.10)

Exercise 2.2.4. Prove that coherent states defined in 2.10 are normalized.

Exercise 2.2.5. Derive equations 2.10 and 2.11.

Exercise 2.2.6. Repeat the derivations in section 2.2.2.1 to obtain the momentum distribution $p(p)$ related to $p(x)$

2.2.3 *Generation of a coherent state*

2.3 Squeezed states

Another class of important states are squeeze states. We shall consider a simple mode to simplify the discussion. To begin with we consider two hermitian operators A and B and define the variance of an operator $\sigma(A) = (\Delta A)^2 =: \langle A^2\rangle - \langle A\rangle^2$, where the average is over some quantum state. With these definitions it is possible to formulate an uncertainty principle as [Robertson (1929)]

$$\Delta A \Delta B \geq \frac{1}{2}\Big|[A, B]\Big| \tag{2.18}$$

(✍ **Exercise 2.3.1**)

In the case of an harmonic oscillator ($m = 1$) and for a coherent state we have $(\Delta x)^2 = \frac{\hbar}{2\omega}$ and $(\Delta p)^2 = \frac{\hbar\omega}{2}$ and from equation (2.18) $\Delta x\Delta p \geq \frac{1}{2}|\langle[x, p]\rangle| = \hbar/2$. Hence, as stated before coherent states are minimum uncertainty states.

At this stage it is convenient to define the *quadratures* of the EM-field $X_1 = a+a^\dagger$ and $X_2 = -i(a - a^\dagger)$ satisfying $[X_1, X_2] = 2i$. Many observable quantities of the EM-field depend on them. If we use equation (2.18) it follows

$$\Delta X_1 \Delta X_2 \geq 1. \tag{2.19}$$

From equation (2.19) it is possible to classify the set of EM-field states according to their quadrature variances as follows:

- *Minimum uncertainty states*, those for which $\Delta X_1\Delta X_2 = 1$. In the space $(\Delta X_1\Delta X_2)$ they define an hyperbola and remembering that
$$x = \sqrt{\frac{\hbar}{2\omega}}(a^\dagger + a) \quad \text{and} \quad p = i\sqrt{\frac{\hbar\omega}{2}}(a - a^\dagger) \tag{2.20}$$
we conclude that for a *coherent state* $\Delta x\Delta p \geq \frac{1}{2}|\langle[x, p]\rangle| = \hbar/2$. Furthermore we already know that $(\Delta x)^2 = \frac{\hbar}{2\omega}$ and $(\Delta p)^2 = \frac{\hbar\omega}{2}$, and therefore coherent states fulfill that $\Delta X_1 = \Delta X_2 = 1$.
- *Ideal squeeze states* They are minimum uncertainty states (so $\Delta X_1\Delta X_2 = 1$) for which $\Delta X_1 < 1$ or $\Delta X_2 < 1$.
- *Squeeze states* The are such that $\Delta X_1 < 1$ or $\Delta X_2 < 1$ but $\Delta X_1\Delta X_2 > 1$ In figure 2.2 they are represented in the $(\Delta X_1, \Delta X_2 < 1)$ space.

2.3.1 *Quadratures for a coherent state*

The quadratures X_1 and X_2 we have introduced can be easily computed for a coherent state, they are explicitly written as

$$\begin{aligned} &\langle\alpha|X_1|\alpha\rangle = \langle\alpha|a + a^\dagger|\alpha\rangle) = \alpha + \alpha^* = 2Re(\alpha) \\ &\langle\alpha|X_2|\alpha\rangle = -i\langle\alpha|a - a^\dagger|\alpha\rangle) = -i(\alpha - \alpha^*) = 2Im(\alpha) \quad \text{and} \\ &(\Delta X_1)^2 = 1. \end{aligned} \tag{2.21}$$

(✍ **Exercise 2.3.2**)

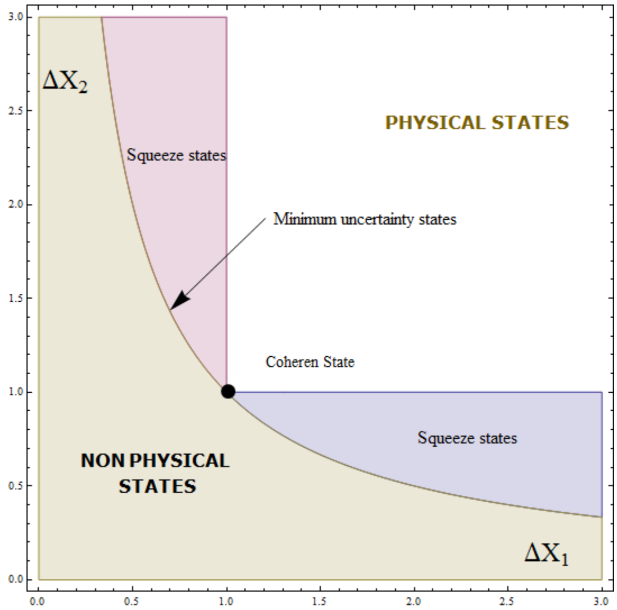

Fig. 2.1

2.3.2 *How are squeeze states?*

We shall define first the *squeeze* operator

$$S(g) =: \exp\left(\frac{g^*}{2}a^2 - \frac{g}{2}a^{\dagger 2}\right) \quad \text{with} \quad g \in \mathbb{C}, \tag{2.22}$$

$g = |g|e^{i\phi} =: re^{i\phi}$, with $r, \phi \in \mathbb{R}$. The squeeze operator satisfies $S^\dagger(g) = S^{-1}(g) = S(-g)$, (✍ **Exercise 2.3.3**). In addition the action of $S(g)$ on the annihilation and creation operators is given by

$$S^\dagger(g)aS(g) = a\cosh r - a^\dagger e^{-2i\phi}\sinh r$$
$$S^\dagger(g)a^\dagger S(g) = a^\dagger \cosh r - ae^{2i\phi}\sinh r$$

and for the quadratures,

$$S^\dagger(g)(X_1 + iX_2)e^{-i\phi}S(g) = e^{-r}e^{-i\phi}X_1 + ie^r e^{-i\phi}X_2. \tag{2.23}$$

The action of the squeeze operator is to attenuate one quadrature of the field and to magnify the other. Usually r is named *squeeze factor*.

It is possible to define a squeeze state starting from vacuum as

$$|\alpha g\rangle =: D(\alpha)S(g)|0\rangle. \tag{2.24}$$

So defined squeezed states satisfy the following properties:

Property 1. $\langle X_1 + iX_2\rangle = \langle 2a\rangle = 2\alpha$
Property 2. $\langle \alpha g|X_1 - iX_2|\alpha g\rangle = \langle \alpha|X_1 - iX_2|\alpha\rangle$

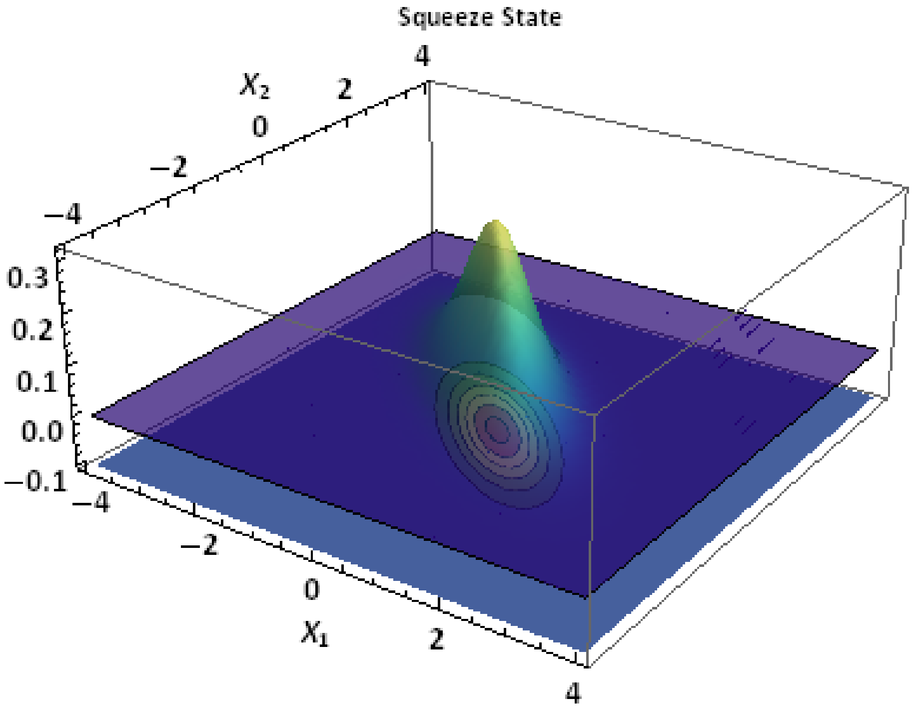

Fig. 2.2 Squeeze state.

Property 3. $\Delta(X_1 e^{-i\phi}) = e^{-r}$
Property 4. $\Delta(iX_2 e^{-i\phi}) = e^{r}$
Property 5. $\langle N = \text{photon number}\rangle = |\alpha|^2 + \sinh^2 r$
Property 6. $(\Delta N)^2 = |\alpha \cosh r - \alpha^* e^{2i\phi} \sinh r|^2 + 2\cosh^2 r \sinh^2 r$ (✍ **Exercise 2.3.4**)

It is instructive to treat a simple harmonic oscillator which it is initially in an squeezed state and to compute the mean position and variance. A straight forward calculation shows that

$$\langle X(t)\rangle = \sqrt{\frac{2\hbar}{\omega}} \cos\omega t \quad \text{and}$$
$$(\Delta X)^2 = \frac{\hbar}{2\omega}\big(\cosh^2 r + \sinh^2 r - 2\cosh r \sinh r \cos(2\omega t - 2\phi)\big), \tag{2.25}$$

which implies that while X oscillates with frequency ω, its variance does it at frequency 2ω. Thus the probability density is *breathing* during its time evolution (FIGURE). (✍ **Exercise 2.3.5**)

2.3.3 *Generation of squeeze states*

The squeeze operator defined in equation (2.22) has a physical realization through the hamiltonian

$$H = \frac{\hbar}{2}\left(\chi(a^\dagger)^2 + \chi^* a^2\right), \tag{2.26}$$

which describes simultaneous two-photon generation or absorption processes. These processes can be realized by second order nonlinear processes where a photon of energy $2\hbar\omega$ generates two photons of energy $\hbar\omega$.

(✍ **Exercise 2.3.6**)

(✷— **Project 2.3.7**)

Exercises

Exercise 2.3.1. Prove equation 2.18 and discuss under which conditions is valid.

Exercise 2.3.2. Derive al equalities in 2.21.

Exercise 2.3.3. Check this statements.

Exercise 2.3.4. Prove properties 3.1

Exercise 2.3.5. Derive the position expectation value and position variance for a harmonic oscillator which is initially in an squeezed state. See equations (2.25)

Exercise 2.3.6. prepare a little review of the process describe by the hamiltonian 2.26

Exercise 2.3.7. **Project**: One application of squeeze light is in detection of gravitational radiation. Start a little project by reading the references [Walls and Milburn (2008)] (section 8.3.1) and references [Caves (1980, 1981)]

2.4 Fluctuations of the Electric field

From different states it is possible to compute the expectation values of the field. However useful physical information is contained in the field fluctuations. In this section we study fluctuations of a single mode confined in a box with volume $V = L^3$ with the form

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= i\boldsymbol{e}_\lambda\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(a(0)e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})} - cc\right)\\ &= \boldsymbol{e}_\lambda\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}(X_1\sin(\omega t-\boldsymbol{k}\cdot\boldsymbol{r}) - X_2\cos(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})),\end{aligned} \tag{2.27}$$

from which follows

$$\langle\boldsymbol{E}(\boldsymbol{r},t)\rangle = \boldsymbol{e}_\lambda\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}(\langle X_1\rangle\sin(\omega t-\boldsymbol{k}\cdot\boldsymbol{r}) - \langle X_2\rangle\cos(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})). \tag{2.28}$$

The uncertainty of the field is defined as

$$
\begin{aligned}
(\Delta \boldsymbol{E})^2 &=: \sigma(\boldsymbol{E}) = \langle \boldsymbol{E}(\boldsymbol{r},t)^2\rangle - \langle \boldsymbol{E}(\boldsymbol{r},t)\rangle^2 \\
&= \frac{\hbar\omega}{2\epsilon_0 V}\Big(\sigma(X_1,X_1)\sin^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) + \sigma(X_2,X_2)\cos^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) \\
&- \sigma(X_1,X_2)\sin(2\omega t - 2\boldsymbol{k}\cdot\boldsymbol{r})\Big),
\end{aligned}
\tag{2.29}
$$

with the definition $\sigma(A,B) =: \frac{1}{2}\langle AB+BA\rangle - \langle A\rangle\langle B\rangle$. In fact for a minimum uncertainty state $\Delta X_1 \Delta X_2 = 1$ it follows $\sigma(X_1,X_2)=0$,*statement* (2.4)

(✍ **Exercise 2.4.1**).
(✍ **Exercise 2.4.2**).
(✍ **Exercise 2.4.3**)

Exercises

Exercise 2.4.1. Prove statement 2.4

Exercise 2.4.2. Derive the $\langle \boldsymbol{E}\rangle$ and $\sigma(\boldsymbol{E})$ for a coherent state $|\alpha\rangle$

Exercise 2.4.3. Derive the $\langle \boldsymbol{E}\rangle$ and $\sigma(\boldsymbol{E})$ for an ideal squeeze state.
In exercise (2.4.3) consider two cases

(1) $\phi = 0$ and $r > 0$
(2) $\phi = 0$ and $r < 0$

2.5 Multimode squeezed states

We have discussed single mode squeezed states in the previous section, but it is possible to generate multimode squeezed states. They can be experimentally realized by means of a nondegenerate parametric amplifier in which two photons of frequency $\omega_1+\omega_2$ are generated from an input signal of frequency ω. The interaction Hamiltonian if

$$
H_I = \hbar(\chi a_1^\dagger a_2^\dagger - \chi^* a_1 a_2), \tag{2.30}
$$

where χ is a nonlinear coupling coefficient proportional to the nonlinear susceptibility of the medium and to the amplitude of the pump field of frequency ω. A squeezed state $|\alpha_1,\alpha_2\rangle$ for the system is generated from vacuum. First a squeeze operation is apply to vacuum and after both modes are displaced *i.e.*

$$
|\alpha_1,\alpha_2\rangle = D_1(\alpha_1)D_2(\alpha_2)S(\xi)|0\rangle, \tag{2.31}
$$

with the definitions

$$
\begin{aligned}
D_i(\alpha) &= e^{\alpha a_i^\dagger - \alpha^* a_i} \quad (i=1,2) \quad \text{and the two mode squeezing operator} \\
S(\chi) &= e^{\chi^* a_1 a_2 - \chi a_1^{\dagger 2} a_1^{\dagger 2}} \quad \text{with} \quad \chi = re^{i\phi}.
\end{aligned}
\tag{2.32}
$$

The two mode squeeze operator acts on the individual modes annihilation operators as

$$S^\dagger(\chi) a_j S(\chi) = a_j \cosh(r) - a_j^\dagger \sinh(r) e^{i\phi}, \quad \text{with} \quad j = 1, 2. \tag{2.33}$$

If $\theta = 0, \chi = r \in \mathbb{R}$ the corresponding state has the following expansion in term of Fock states

$$S(\chi)|0\rangle = \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} \tanh^n(r) |n\, n\rangle. \tag{2.34}$$

(✍ **Exercise 2.6.1**)

The state (2.34) is an example of entangled state as such state cannot be written as a product state of the form $|\psi\rangle = |\psi_1\rangle \bigotimes |\psi_2\rangle$. If the photon number is measured (Von Neumann measurement) for mode 1, with result n then it is known with certainty the phonon number of mode 2, notice that the state (2.34) is diagonal in the Fock basis.

2.6 Quadrature correlation

As an example of quadrature correlations for two mode squeeze states we pay attention to a physical realization of such state. In parametric down conversion it happens than one high energy photon (classical pump) with energy $2\hbar\omega = \hbar\omega_1 + \hbar\omega_2$ splits in two photons of energies $\hbar\omega_1$ and $\hbar\omega_2$, when it interacts with a nonlinear media. The two resulting photons are usually refer as the *signal* and the *idler* photon. In such phenomena it is realized a squeeze state. The Hamiltonian describing this situation is

$$H = \hbar\omega_1 \underbrace{a_1^\dagger a_1}_{n_1} + \hbar\omega_2 \underbrace{a_2^\dagger a_2}_{n_2} + i\hbar g (a_1^\dagger a_2^\dagger e^{-2i\omega t} - a_1 a_2 e^{2i\omega t}), \tag{2.35}$$

where $a_1 (a_2)$ refer to the signal and idler photons. The coupling constant g is proportional to the secnd-order susceptibility of the nonlinear media and the amplitude of the classical pump field [Walls and Milburn (2008)]. The photons are generated simultaneously as infered by the fact that

$$[n_1 - n_2, H] = 0, \tag{2.36}$$

and therefore $n_1(t) - n_2(t) = n_1(0) - n_2(0)$. The Heisenberg equations of motion in the interaction picture for a_1 and $a_2^\dagger$ are

$$\begin{aligned} \dot{a}_1(t) &= g\, a_2^\dagger \\ \dot{a}_2^\dagger(t) &= g\, a_1. \end{aligned} \tag{2.37}$$

(✍ **Exercise 2.6.2**)

The solution of the above equations are

$$a_1(t) = a_1(0)\cosh gt + a_2^\dagger \sinh gt$$
$$a_2(t) = a_2(0)\cosh gt + a_1^\dagger \sinh gt. \tag{2.38}$$

With this solution at hand let's workout some quantum correlations explicitly. Consider that initially the system is in a coherent state $|\alpha_1\alpha_2\rangle$ for which the mean photon number in the mode 1 is

$$\langle n_1(t)\rangle = \langle\alpha_1\alpha_2|n_1(t)|\alpha_1\alpha_2\rangle = |\alpha_1\cosh gt + \alpha_2^*\sinh gt|^2 + \sinh^2 gt. \tag{2.39}$$

☞Notice that the last term in equation (2.39) remains when the initial state is $\alpha_1 = \alpha_2 = 0$ and can be interpreted as amplification of vacuum fluctuation at a given time t due to the pump field and its interaction with the nonlinear media.

As a matter of fact intensity correlations in this system have interesting quantum features that it is woth discussing. In particular, quantum correlations of the two modes may violate classical inequalities, as we shall see in the following. As a first example lets look carefully to the expectation value $\langle n_1 n_2\rangle$ which can be expressed by means of the Glauber-Sudarshan P function [Walls and Milburn (2008)] as

$$\langle n_1 n_2\rangle = \int d^2\alpha_1 \int d^2\alpha_2 |\alpha_1|^2|\alpha_2|^2 P(\alpha_1,\alpha_2). \tag{2.40}$$
$$\text{with}\quad d^2\alpha = d(\mathrm{Re}\,\alpha)d(\mathrm{Im}\,\alpha)$$

It is important to emphasize that if a *positive* P function exists equation 2.41 corresponds to a *classical intensity* correlation function for the fields 1 and 2 with amplitudes α_1 and α_2. In such conditions we may use the Hölder inequality (see http://en.wikipedia.org/wiki/H%C3%B6lder_inequality

$$\int_{S_1}\int_{S_2} |f(x,y)\,g(x,y)|\,\mu_2(\mathrm{d}y)\,\mu_1(\mathrm{d}x) \tag{2.41}$$
$$\leq \left(\int_{S_1}\int_{S_2} |f(x,y)|^p\,\mu_2(\mathrm{d}y)\,\mu_1(\mathrm{d}x)\right)^{1/p}\left(\int_{S_1}\int_{S_2} |g(x,y)|^q\,\mu_2(\mathrm{d}y)\,\mu_1(\mathrm{d}x)\right)^{1/q}.$$

$$\left(\text{with}\quad 1 \leq p,q \leq \infty \quad \text{and} \quad 1 = \frac{1}{p}+\frac{1}{q}\right)$$

to write

$$\int d^2\alpha_1 \int d^2\alpha_2 |\alpha_1|^2|\alpha_2|^2 P(\alpha_1,\alpha_2) \leq \left(\int d^2\alpha_1|\alpha_1|^4 P(\alpha_1,\alpha_2)\right)^{1/2}$$
$$\times\left(\int d^2\alpha_2|\alpha_2|^4 P(\alpha_1,\alpha_2)\right)^{1/2} \tag{2.42}$$
$$\tag{2.43}$$

or equivalently

$$\langle n_1 n_2\rangle = \langle a_1^\dagger a_1 a_2^\dagger a_2\rangle \leq \left(\langle (a_1^\dagger)^2 a_1^2\rangle\langle (a_2^\dagger)^2 a_2^2\rangle\right)^{\frac{1}{2}}. \tag{2.44}$$

In the case of symmetric modes last equation leads to the Cauchy-Schwartz inequality

$$\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle \le \langle (a_1^\dagger)^2 a_1^2\rangle. \tag{2.45}$$

In this last set of equations it has been assumed that the P function is positive, therefore there are quantum states for which inequality (2.45) is violated. It is convenient to define the second-order intensity correlation function for a single-mode as

$$g_i^{(2)}(0) = \frac{\langle a_i^\dagger a_i^\dagger a_i a_i\rangle}{\langle a_i^\dagger a_i\rangle^2} \quad \text{with} \quad i = 1, 2, \tag{2.46}$$

and for a two-mode field

$$g_{12}^{(2)}(0) = \frac{\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle}{\langle a_1^\dagger a_1\rangle\langle a_2^\dagger a_2\rangle}, \tag{2.47}$$

in terms of which 2.42 reads

$$(g_{12}^{(2)}(0))^2 \le g_1^{(2)}(0) g_2^{(2)}(0). \tag{2.48}$$

In a more general case, states for which the P function not need to be positive, a more general inequality may be derived. Consider two state vectors $|f\rangle$ and $|g\rangle$ in the Hilbert space of the two-modes system. The Hilbert space has an inner product denoted by $\langle f|g\rangle$. In this setting the Cauchy-Schwartz inequality is $\langle f|g\rangle^2 \le \langle f|f\rangle\langle g|g\rangle$ that applied in our context leads to

$$\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle^2 \le \langle (a_1^\dagger a_1)^2\rangle\langle (a_2^\dagger a_2)^2\rangle, \tag{2.49}$$

that for symmetric modes is

$$\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle \le \langle (a_1^\dagger a_1)^2\rangle = \langle (a_1^\dagger)^2 a_1^2\rangle + \langle a_1^\dagger a_1\rangle \tag{2.50}$$

that in terms of second-order intensity correlations reads

$$\boxed{g_{12}^{(2)}(0) \le g_1^{(2)}(0) + \frac{1}{\langle a_1^\dagger a_1\rangle}.} \tag{2.51}$$

(✍ **Exercise 2.6.3**)

As previously noticed there is a conservation law in the system *i.e.* $n_1(t) - n_2(t) = n_1(0) - n_2(0)$ from which follows

$$\langle n_1(t) n_2(t)\rangle = \langle n_1^2(t)\rangle + \langle n_1(t)(n_2(0) - n_1(0))\rangle. \tag{2.52}$$

If the averages are performed with respect to vacuum state then

$$\langle n_1(t) n_2(t)\rangle = \langle a_1^\dagger(t) a_1^\dagger(t) a_1(t) a_1(t)\rangle + \langle a_1^\dagger(t) a_1(t)\rangle, \tag{2.53}$$

which means that in this situation equation (2.50) fulfills the equality, what implies a maximum violation of equation (2.50) that quantum mechanics allows. *In fact this system shows quantum correlations that systematically violate classical inequalities.*

Exercises

Exercise 2.6.1. Prove equation (2.34).

Exercise 2.6.2. Derive and solve this set of equations.

Exercise 2.6.3. Reproduce all steps to arrive to equation (2.50).

2.6.1 *Einstein-Podolsky-Rosen (EPR)*

Let's discuss the EPR paradox in the context of two-mode correlated states as those discussed so far. We begin with some definition as

$$X_j^\varphi =: a_1(t)e^{i\omega_j t}e^{i\varphi} + a_1^\dagger(t)e^{-i\omega_j t}e^{-i\varphi}. \tag{2.54}$$

With this definition follows that

$$[X_j^\varphi, X_j^{\varphi+\frac{\pi}{2}}] = -2i, \tag{2.55}$$

which suggests that X_j^φ and $X_j^{\varphi+\frac{\pi}{2}}$ are analogous to position an momentum operators. The interest is to measure the degree of correlation that the two modes have. To that end we consider

$$V(\varphi,\psi) =: \frac{1}{2}\langle (X_1^\varphi - X_2^\psi)^2\rangle. \tag{2.56}$$

If $V(\varphi,\psi) = 0$ then there is a perfect correlation between the two modes, in the sense that from a measurement of X_1^φ it is possible to infer the value of X_2^ψ with certainty. To understand this statement consider $|\xi_1\rangle$ and $|\xi_2\rangle$ eigenvectors of X_1^φ and X_2^ψ respectively with eigenvalues ξ_1 and ξ_2, then if

$$\begin{aligned} 0 = V(\varphi,\psi) &= \langle\Psi|(X_1^\varphi - X_2^\psi)^2|\Psi\rangle \\ &= \frac{1}{2}\sum_{\xi_1\xi_2}\langle\Psi|\xi_1\xi_2\rangle\langle\xi_1\xi_2|(X_1^\varphi - X_2^\psi)^2|\Psi\rangle \\ &= \frac{1}{2}\sum_{\xi_1\xi_2}\overbrace{p_{\xi_1\xi_2}}^{=|\langle\Psi|\xi_1\xi_2\rangle|^2}(\xi_1-\xi_2)^2, \end{aligned} \tag{2.57}$$

with $p_{\xi_1\xi_2} > 0$ then it must happen that $\xi_1 = \xi_2$ to (2.57) be true. In this sense if the two observables are perfectly correlated the result ξ_1 of a measurement of X_1^φ implies with certainty the result ξ_2 when measuring X_2^ψ. We consider now a situation in which many two-mode equal states are initially prepared and are measured the quantities X_2^ψ and $X_2^{\psi+\frac{\pi}{2}}$. The variances of both quadratures satisfy the uncertainty relation

$$\sigma(X_2^\psi)\sigma(X_2^{\psi+\frac{\pi}{2}}) \geq 1. \tag{2.58}$$

But the system shows *perfect* correlations between X_2^ψ and X_1^φ and $X_2^{\psi+\frac{\pi}{2}}$ and $X_1^{\varphi-\frac{\pi}{2}}$, so we can measure X_1^φ and infer from the result the value of X_2^ψ. In a real setting no perfect measurement can be made, some error is always made. We denote by $\sigma_{inf}(X_2^{\psi})$ the variance that characterizes the error made in the estimation of X_2^{ψ} by measuring X_1^{φ}. In a similar manner is defined the variance $\sigma_{inf}(X_2^{\psi+\frac{\pi}{2}})$. It turns out that such variances are not constrained by equation (2.58). In this sense it is said that there is a EPR correlation that is paradoxical because it is not constrained by the Heisenberg. However there is not contradiction as in the first case the variances entering in equation (2.58) are evaluated in the same state. On the other hand the inferred variance $\sigma_{inf}(X_2^{\psi})$ is computed in the conditional state given a result from a measurement on X_2^{ψ} and the variance $\sigma_{inf}(X_2^{\psi+\frac{\pi}{2}})$ is evaluated at a different state which is the conditional state given a result from a measurement on $X_2^{\psi+\frac{\pi}{2}}$. Therefore being computed in different states not need to be constrained by the Uncertainty principle.

(✷— **Project 2.6.4**)

Exercises

Exercise 2.6.4. **Project**: The discussion made in this section can be completed with a more deep reading of the following references [Walls and Milburn (2008)] (section 5.2.3) and [Reid (1989)]. An experimental measurement of the inferred variances in a parametric amplifier can be found in [Ou and Mandel (1988)]

Chapter 3

Coherence properties

In this chapter we shall discuss what properties of the EM-field are measured.

3.1 Field correlation functions

Suppose we have a detector at position $\boldsymbol{r}$ that at time t is able to detect the $\boldsymbol{E}^{+}(\boldsymbol{r},t) \propto a$ component of the EM-field by means of an annihilation of a photon. The transition rate associated to the absorption of a photon in the detector can be written using the Fermi-Golden rule, that for the initial and final states $|i\rangle|f\rangle$ is given by

$$T_{i\to f} = \big|\langle f|\boldsymbol{E}^{+}(\boldsymbol{r},t)|i\rangle\big|^{2}. \tag{3.1}$$

If we are interested in the total count rate and not in the particular final state of the field the we can some up contributions over all final states to get the whole intensity *i.e.*

$$I(\boldsymbol{r},t) = \sum_{f} T_{i\to f} = \langle i|\boldsymbol{E}^{-}(\boldsymbol{r},t)\boldsymbol{E}^{+}(\boldsymbol{r},t)|i\rangle. \tag{3.2}$$

In many physical situations we are dealing with mixtures of initial states which are described by density matrix ρ therefore

$$I(\boldsymbol{r},t) = \mathrm{Tr}\big(\rho\boldsymbol{E}^{-}(\boldsymbol{r},t)\boldsymbol{E}^{+}(\boldsymbol{r},t)\big) \tag{3.3}$$

A simple and illustrative consist in considering a Fock state $|n_{\mathbf{k}}\rangle$ as initial state. Remembering that

$$\boldsymbol{E}^{+}(\boldsymbol{r},t) = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0}}\Big(a_{\mathbf{k}}(0)\boldsymbol{u}_{\mathbf{k}}(\boldsymbol{r})e^{-i\omega_{\mathbf{k}}t}\Big), \tag{3.4}$$

with $\boldsymbol{E}^{-}(\boldsymbol{r},t) = (\boldsymbol{E}^{+}(\boldsymbol{r},t))^{\dagger}$, from which

$$I(\boldsymbol{r},t) = \langle n_{\mathbf{k}}|\boldsymbol{E}^{-}(\boldsymbol{r},t)\boldsymbol{E}^{+}(\boldsymbol{r},t)|n_{\mathbf{k}}\rangle = \frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0}n_{\mathbf{k}}\big|\boldsymbol{u}_{\mathbf{k}}(\boldsymbol{r})\big|^{2}. \tag{3.5}$$

We can see that the intensity $I(\boldsymbol{r},t)$ involves the correlation $\mathrm{Tr}\big(\rho\boldsymbol{E}^{-}(\boldsymbol{r},t)\boldsymbol{E}^{+}(\boldsymbol{r},t)\big)$, this motivates the definition of the *n-th order field correlation function* (or *count rate of a n-photon detector*),

$$\boxed{\begin{aligned}&G^{(n)}(x_1, x_2, \cdots, x_n, x'_{n+1}, x'_{n+2}, \cdots, x'_{2n}) =: \\ &\mathrm{Tr}\Big(\boldsymbol{E}^-(\boldsymbol{r}_1, t_1)\cdots \boldsymbol{E}^-(\overbrace{\boldsymbol{r}_n, t_n}^{=:x_n})\boldsymbol{E}^+(\underbrace{\boldsymbol{r}_{n+1}, t_{n+1}}_{=:x'_{n+1}})\cdots \boldsymbol{E}^+(\boldsymbol{r}_{2n}, t_{2n})\Big).\end{aligned}}$$

Notice that $I(\boldsymbol{r}, t) = G^{(1)}(x, x)$. The field correlation functions do satisfy important properties.

Property 1. $G^{(n)}(x_1, \cdots, x_n, x_n, \cdots, x_1) \geq 0$. (✍ **Exercise 3.4.1**).

Property 2. If $A = \sum_{j=1}^n \lambda_j E_j^+(x_j)$ the it follows $\sum_{j,l=1}^n = \lambda_j^* \lambda_l G^{(1)}(x_j, x_l) \geq 0$, hence the matrix defined by $(\boldsymbol{G})_{il} := G^{(1)}(x_j, x_l)$ forms a semi-definite positive quadratic form with the property

$$\det\big|\boldsymbol{G}\big| \geq 0. \tag{3.6}$$

In particular for $n = 2$ it is founded

$$G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2) \geq \left|G^{(1)}(x_1, x_2)\right|^2. \tag{3.7}$$

3.2 Optical coherence

We shall see in this section that in the double slit experiment it is measured the first order correlation function: *i.e. intensities.*

Consider a double slit set up as the one shown in Fig. **??**. Then a plane wave with wave vector $\boldsymbol{k}$ hits the double slit. In the two slits holes, located at positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. At those holes are form two spherical $E_1^+(\boldsymbol{r}, t), E_2^+(\boldsymbol{r}, t)$ waves and therefore the Em-field at the slit, located at $\boldsymbol{r}$ the positive component of the field is

$$\boldsymbol{E}^+(\boldsymbol{r}, t) = E_1^+(\boldsymbol{r}, t) + E_2^+(\boldsymbol{r}, t). \tag{3.8}$$

For spherical waves $\boldsymbol{u}_{\boldsymbol{k}}(\boldsymbol{r}) = \boldsymbol{e}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}/r\sqrt{4\pi\mathcal{V}}$ (see equaiton 3.4 and your notes on the quantization of the EM-field), where $\boldsymbol{e}_{\boldsymbol{k}}$ is the field polarization and $\mathcal{V}$ is the volumen quantization. If the distance between the double slit plane and the screen is R, $\boldsymbol{s}_i = \boldsymbol{r} - \boldsymbol{r}_i$, $s_i = |\boldsymbol{s}_i|$ and $R \gg s_i$, then the field intensity at the screen is,

$$\begin{aligned} I(\boldsymbol{r}, t) &= G^{(1)}(x_1, x_1) + G^{(1)}(x_2, x_2) + 2\mathrm{Re}G^{(1)}(x_1, x_2) \\ &\propto \langle a_1^\dagger a_1\rangle + \langle a_2^\dagger a_2\rangle + 2\Big|\langle a_1^\dagger a_2\rangle \cos\phi\Big|, \end{aligned} \tag{3.9}$$

with $\phi = k(s_1 - s2)$. Consequently an interference patterns emerges with maximum intensity at $\phi = 2\pi n$ $n = 0, 1, 2, \cdots$ (✍ **Exercise 3.4.2**). Let us look at some examples.

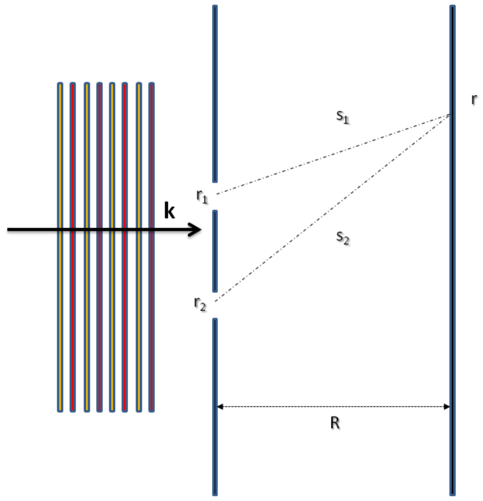

Fig. 3.1 Double slit experiment.

3.2.1 *Examples*

- *Coheren state.* For a coherent state $|\alpha\rangle$ it follows
$$I(\boldsymbol{r},t) \propto 2|\alpha|^2(1+\cos\phi). \tag{3.10}$$
- *A single photon* $|i\rangle = (a_1^\dagger + a_2^\dagger)|0\rangle/\sqrt{2}$
$$I(\boldsymbol{r},t) \propto (1+\cos\phi). \tag{3.11}$$
- *A plane wave* $|i\rangle = (a_1^\dagger a_2^\dagger)|0\rangle$ No interference is observed. (✍ **Exercise 3.4.3**).

The interference pattern is due to $G^{(1)}(x_1, x_2)$, however this correlation is not able to distinguish classical from quantum light. In the next sections we will examine this point.

3.3 First order coherence

Let us define the visibility ν as
$$\begin{aligned}\nu &= \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \\ &= \frac{G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2) + 2\mathrm{Re}G^{(1)}(x_1,x_2) - (G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2) - 2\mathrm{Re}G^{(1)}(x_1,x_2))}{2(G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2))} \\ &= \frac{|G^{(1)}(x_1,x_2)|}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}}\frac{2\sqrt{I_1 I_2}}{I_1 + I_2}.\end{aligned} \tag{3.12}$$

The previous expression suggest the definition

$$g^{(1)}(x_1,x_2) = \frac{|G^{(1)}(x_1,x_2)|}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}} \leq 1. \quad \text{(See equation 3.7)} \tag{3.13}$$

The larger is $g^{(1)}$ more coherent is the field.

3.4 Photon correlation measurements. Second order coherence

The second order coherence function is defined as

$$\begin{aligned} G^{(2)}(\tau) &= \langle E^-(\boldsymbol{r},t)E^-(\boldsymbol{r},t+\tau)E^+(\boldsymbol{r},t+\tau)E^+(\boldsymbol{r},t)\rangle \\ &= \langle : I(t)I(t+\tau) :\rangle \propto \langle : n(t)n(t+\tau) :\rangle, \end{aligned} \tag{3.14}$$

where $: * :$ indicates normal ordering in a expression, in which all $a^\dagger$ operators appear to the left of the expression and the a operators to the right, for example $: aa^\dagger := a^\dagger a + 1$. The $g^{(2)}(\tau)$ function is written as

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{(G^{(1)}(0))^2} = \frac{\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle}{\langle a^\dagger(0)a(0)\rangle^2}. \tag{3.15}$$

(✍ **Exercise 3.4.4**)

Exercises

Exercise 3.4.1. Prove property Property 1.. First notice that $\mathrm{Tr}(\rho A^\dagger A) \geq 0$, and then take $A = \boldsymbol{E}^+(x_1)\cdots\boldsymbol{E}^+(x_n)$

Exercise 3.4.2. Derive equation 3.9

Exercise 3.4.3. Derive all intensities discussed in the examples 3.2.1

Exercise 3.4.4. From the definition of $G^{(2)}$ derive the equality 3.15 and express the result in terms of the variance of n. In addition discuss the result and the bunching and antibunching of ligth.

Chapter 4

P and Wigner distributions

4.1 P-distribution

Coherent states form a overcomplete basis. A state can be expand as a superposition of coherent states. The following representation, introduced by Glauber and Sudarshan, is name the p representation. If ρ is a quantum state its P-representation is given by

$$P(\alpha) := P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \tag{4.1}$$

where $d^2\alpha = \mathrm{d}\ \mathrm{Re}\alpha \mathrm{d}\ \mathrm{Im}\alpha$. Some examples are:

4.1.1 *P-representations of some states*

- *Coherent state* $|\beta\rangle$*:* It follows that for a coherent state $P(\alpha) = \delta^(2)(\alpha - \beta)$
- *Chaotic state.* For a chaotic state $P(\alpha) = e^{-\|alpha|^2/\overline{n}}/\pi\overline{n}$
- *Correlations.* $\langle a^{\dagger n} a^m\rangle = \int d^2 P(\alpha) \quad \alpha^{*n}\alpha^m$
- *Second order correlations* $g^{(2)}(0) = 1 + \frac{\int d^2\alpha P(\alpha)(|\alpha|^2 - \langle|\alpha|^2\rangle)^2}{(\int d^2\alpha P(\alpha)|\alpha|^2)^2}$.
- *Variances* of X_1 and X_2.

$$\sigma(X_1) = 1 + \int d^2\alpha P(\alpha)((\alpha + \alpha^*) - (\langle\alpha\rangle + \langle\alpha^*\rangle))$$
$$\sigma(X_2) = 1 - i\int d^2\alpha P(\alpha)((\alpha - \alpha^*) - (\langle\alpha\rangle - \langle\alpha^*\rangle)).$$

(✍ **Exercise 4.2.1**)

An important point to notice in the previous expressions is that in the case one of the variances were smaller that one this necessarily implies that $P(\alpha)$ should be negative for some α. This shows that $P(\alpha)$ cannot be taken as a true probability distribution. If this is the case the we may think on ρ as a truly quantum state showing features beyond classical ones.

4.2 Wigner function

The characteristic function $\xi(\eta)$ is defined as

$$\xi(\eta) := \mathrm{Tr}(\rho e^{\eta a^\dagger - \eta^* a}), \tag{4.2}$$

and from it the Wigner function for the state ρ is obtained by a Fourier transform, *i.e.*

$$\begin{aligned} W(\alpha) : &= \frac{1}{\pi^2}\int \xi(\eta) e^{\eta\alpha^* - \eta^*\alpha} d^2\,\eta, \\ &= \frac{2}{\pi}\int P(\beta) e^{|\beta-\alpha|^2} d^2\beta. \end{aligned} \tag{4.3}$$

As it is defined the Wigner functions has the following properties

Property 1. It is normalized. $\int W(\alpha) d^2\alpha = 1$.
Property 2. If $\alpha = (x_1 + ix_2)/2$ and $W(x_1, x_2) = W(\alpha)/4)$ then the following relations hold (marginals)

$$\begin{aligned} P(x_1) &= \int dx_2 W(x_1, x_2) \\ P(x_2) &= \int dx_1 W(x_1, x_2) \end{aligned} \tag{4.4}$$

For different states it is possible to have explicit forms of the Wigner functions.

4.2.1 *Examples of Wigner functions*

- *Coherent state.* If $|\alpha\rangle = |(X_1 + iX_2)/2\rangle$ then
$$W(x_1, x_2) = e^{-(x_1-X_1)^2-(x_2-X_2)^2}/2\pi.$$
- *Squeezed state.*
$$W(x_1, x_2) = e^{-(x_1-X_1)^2e^{2r}-(x_2-X_2)^2e^{-2r}}/2\pi.$$
- *Number state.* For a number state $|n\rangle$ the Wigner function is given by
$$W(x_1, x_2) = \frac{2}{\pi}(-1)^2 L_n(4r^2) e^{-2r^2}, \tag{4.5}$$
with $r^2 = x_1^2 + x_2^2$, and $L_n(x)$ the Laguerre polynomial. Notice that this Wigner function is clearly negative in some domain.

(✍ **Exercise 4.2.2**)

Exercises

Exercise 4.2.1. Prove all relations enumerated in 4.1.1.

Exercise 4.2.2. Prove all relations enumerated in 4.2.1.

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